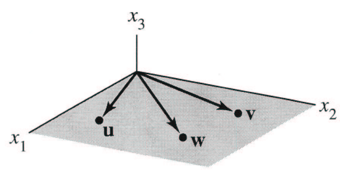
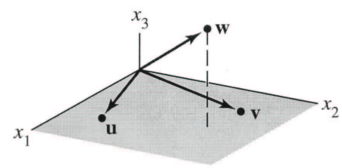
More on linear independence

Last class we saw that a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ is linearly dependent if there is a nontrivial dependence relation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0}$ among them. Otherwise the set is linearly independent.



Linearly dependent,
w in Span{**u**, **v**}



Linearly independent,
w *not* in Span{**u**, **v**}

Example. We know that the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$$

is linearly dependent. What are all possible dependence relations among this set of vectors?

Theorem. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.

Theorem. A nonzero set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors is linearly dependent if and only if, for some index j, the vector $\mathbf{v}_j$ is a linear combination of the vectors $\mathbf{v}_1$, …, $\mathbf{v}_{j-1}$.

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Linear transformations

In order to understand the definition of a linear transformation, let's start with some examples of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. (As we shall see, **not all of these examples are linear transformations.**)

Examples: functions $f : \mathbb{R} \to \mathbb{R}$

1. $f_1(x) = 2x$
2. $f_2(x) = 2x + 1$
3. $f_3(x) = x^2$
4. $f_4(x) = \cos x$

Examples: functions $g : \mathbb{R}^2 \to \mathbb{R}^2$

1. $g_1(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$
2. $g_2(x_1, x_2) = (\cos(x_1 + x_2), x_1 + x_2^2)$

Examples: functions h defined on $\mathbb{R}^3$

1. $h_1(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_2 + x_3)$
2. $h_2(x_1, x_2, x_3) = \dfrac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Definition. Given a function (transformation) $T : \mathbb{R}^n \to \mathbb{R}^m$, we say that T is *linear* if

1. $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, and
2. $T(r\mathbf{v}) = rT(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$ and all r in $\mathbb{R}$.

Terminology: Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$:

- $\mathbb{R}^n$ is the *domain* of T.
- Our textbook says that $\mathbb{R}^m$ is the *codomain* of T.
- The *image* or *range* of T is the set of vectors

$$\{\mathbf{w} \in \mathbb{R}^m \mid T(\mathbf{v}) = \mathbf{w} \text{ for some } \mathbf{v} \in \mathbb{R}^n\}.$$

Example. Consider the function $T(x_1, x_2, x_3) = (x_1, 0)$. Its domain is $\mathbb{R}^3$, its codomain is $\mathbb{R}^2$, and its image is the x_1-axis in $\mathbb{R}^2$. Note that its image is not $\mathbb{R}^1$.

Basic facts about linear transformations T

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(r_1\mathbf{v}_1 + \ldots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + \ldots + r_kT(\mathbf{v}_k)$

Which of the functions given above are linear?

One way to show that a transformation is linear is to verify the two conditions of linearity directly, but there is an easier way to see that these transformations are linear.

Important class of examples: Given an $m \times n$ matrix $\mathbf{A}$, then we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by the equation

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

We know that T is a linear transformation because the matrix-vector product satisfies the necessary conditions.

Example. Let

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Then

$$\mathbf{H}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$