More on linear transformations

Theorem. Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $m \times n$ matrix. This matrix $\mathbf{A}$ is called the *standard matrix representation* of T.

Why? Let's make two observations using the "standard basis" of $\mathbb{R}^n$.

Definition. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \ldots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Observation 1: If we know the images $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ for all of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then we can calculate $T(\mathbf{x})$ for any $\mathbf{x}$ in $\mathbb{R}^n$.

Observation 2: If

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{array} \right],$$

then

$$\mathbf{A}\mathbf{x} = x_1 T(\mathbf{e}_1) + \ldots + x_n T(\mathbf{e}_n).$$

Using trigonometric identities we can show that rotation of $\mathbb{R}^2$ about the origin by a given angle θ is a linear transformation.

Example. What is the matrix representation of the linear transformation that rotates $\mathbb{R}^2$ by 45° ($\pi/4$ radians) counterclockwise around the origin?

Examples of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$

On the course web page, there is a link to a java program called `Matrix Machine` written by Hu Hohn. It lets you investigate the mapping properties of various 2×2 matrix transformations. Try the following three matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

There are also links to two applets by David Austin at Grand Valley State University. These two applets are particularly useful when you want to understand the mapping properties of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

1. rotations

2. reflections

3. contractions and expansions

4. shears

5. projections

Definitions. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. We say that T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.
2. We say that T is one-to-one if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Examples. Let

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the only solution to $\mathbf{T}(\mathbf{x}) = \mathbf{0}$ is the trivial solution.

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\mathbf{A}$ be its standard matrix representation. Then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $\mathbf{A}$ span $\mathbb{R}^m$, and
2. T is one-to-one if and only if the columns of $\mathbf{A}$ are linearly independent.