Solving systems of linear equations

Consider the system of linear equations
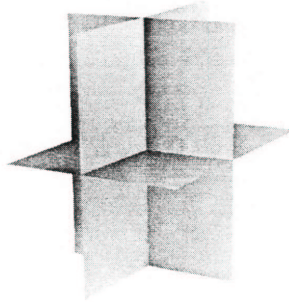
$$2x_1 + x_2 - x_3 = 6$$
$$x_1 + x_2 = 3$$
$$x_1 + x_3 = 1.$$

What does this mean geometrically?

■ **EXAMPLE**   Three equations in three variables. Each equation determines a plane in space.
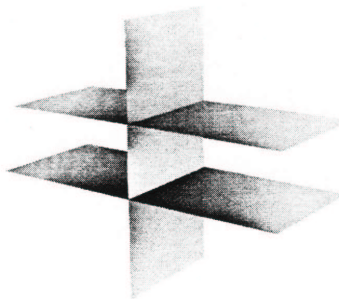
i) The planes intersect in one point:

ii) The planes intersect in a line:

iii) There is no point common to all three planes:

■

1.1.03

Let's do a two-variable example more systematically:

$$3x + y = -2$$
$$-x + 3y = 4$$

Elementary row operations on a matrix

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.

2. (Interchange) Interchange two rows.

3. (Scaling) Replace a row by a nonzero multiple of itself.

Two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other. (Note that each row operation is "reversible" and therefore row equivalence is an equivalence relation.)

Note that the definition of row equivalence has nothing to do with systems of linear equations. It is simply a relationship among matrices of the same size.

Theorem. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

This theorem is not difficult to prove. The only row operation that you may wonder about is row replacement, but if you remember that row operations are reversible, it is not hard to see that row replacement does not change the solution set.

Now let's go back to the system of three linear equations in three unknowns that we discussed at the start of class, but this time we'll be more systematic.

Example. Consider the system of equations

$$2x_1 + x_2 - x_3 = 6$$
$$x_1 + x_2 = 3$$
$$x_1 + x_3 = 1.$$

The augmented matrix associated to this system is

$$\begin{bmatrix} 2 & 1 & -1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Now we use row operations to solve the system of equations:

1. Flip rows R_1 and R_3:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 6 \end{bmatrix}$$

2. Replace row R_2 with $R_2 - R_1$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & -1 & 6 \end{bmatrix}$$

3. Replace row R_3 with $R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -3 & 4 \end{bmatrix}$$

4. Replace row R_3 by $R_3 - R_2$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

At this point we know that the (original) system is consistent and that it has exactly one solution.

We can determine this solution with three more row operations:

5. Replace row R_3 by $(-\frac{1}{2})R_3$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

6. Replace row R_2 by $R_2 + R_3$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

7. Replace row R_1 by $R_1 - R_3$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore, the original system has the unique solution $(x_1, x_2, x_3) = (2, 1, -1)$, which we can check by calculating

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2(2) + (1) - (-1) = 6 \\ x_1 + x_2 &= \quad (2) + (1) \quad = 3 \\ x_1 + x_3 &= \quad (2) + (-1) = 1. \end{aligned}$$

See the course web page for relevant animations.

Row echelon form

A matrix is in **row echelon form** (REF) if it satisfies the following three properties:

1. All nonzero rows are above the zero rows.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zero.

A matrix is in **reduced row echelon form** (RREF) if it is in row echelon form and it satisfies the additional two conditions:

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.

Examples. Consider the following three matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 0 & 3 & 7 & 2 \\ \pi & -1 & 4 & 2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 1. Each matrix is row equivalent to a unique matrix in reduced row echelon form.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

It is row equivalent to the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Suppose that $\mathbf{A}$ is the augmented matrix of a system of linear equations. What is the corresponding solution set?

Some terminology related to RREF

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The matrix **B** determines the pivot positions and the pivot columns of the matrix **A**.

Theorem 2.

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. In other words, a linear system is consistent if and only if the RREF matrix that is row equivalent to the augmented matrix has no row of the form $[0\ 0\ \ldots\ 0\ b]$, where b is nonzero.

2. If a linear system is consistent, then the solution set contains either

   (a) a unique solution, if there are no free variables, or

   (b) infinitely many solutions, where there is at least one free variable.