MA 242 Exams
Fall 2007

Here are the four exams that I gave in my MA 242 class in the Fall of 2007. You should use them to get an idea of the format of a typical test and to see the types of questions I ask. *You should not assume that the test questions this semester will be on the same topics.* In fact, you are always responsible for *all* of the material that we cover in class as well as *all* of the designated material from your text. The best way to study for my exams is to be sure that you are very comfortable with the homework assignments and the examples that I present in class. My tests often vary in difficulty (as you can see here), and your grade for the examination will be determined by a curve that will be announced in class after the examination is graded.

MA 242 Exam 1A October 3, 2007

Name: ______________________________

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 6 pages (not counting this cover page). Please make sure that you have all 6 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
1	18	
2	20	
3	18	
4	16	
5	28	
TOTAL	100	

1. (18 points) Row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -5 & 2 \\ 2 & -3 & -7 & 6 \\ -1 & 3 & 8 & 1 \end{bmatrix}$$

to **reduced row echelon form** (RREF). Do only one row operation at a time and specify that operation when you perform it. Indicate when you first arrive at a matrix in **echelon form** (REF).

2. (20 points) Consider the system

$$\begin{aligned} x_1 + 3x_2 + 2x_4 &= 5 \\ 2x_1 + 6x_2 + x_3 + 2x_4 &= 12 \\ -x_1 - 3x_2 - 2x_4 + x_5 &= -4 \\ 3x_1 + 9x_2 + 6x_4 - 2x_5 &= 13 \end{aligned}$$

of four linear equations in five unknowns. Express its solution set in parametric vector form.

3. (18 points) For each of the following linear transformations T, determine if it is one-to-one and/or onto. **In order to receive any credit, you must provide a brief justification for your answer.**

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$.

4. (16 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points through the x_2-axis and then reflects points through the line $x_2 = x_1$.

   (a) Find the standard matrix representation for T.

   (b) Show that T is merely a rotation about the origin. What is the direction (clockwise or counterclockwise) and the angle of rotation?

5. (28 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

   (a) If the equation $\mathbf{Ax} = \mathbf{b}$ has more than one solution for some $\mathbf{b}$, then the equation $\mathbf{Ax} = \mathbf{0}$ has more than one solution.

   (b) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of vectors in $\mathbb{R}^3$. If none of the vectors in S is a scalar multiple of any of the others, then S is linearly independent.

Question 5 (continued):

(c) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(d) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\mathbf{v}_3$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

MA 242	Exam 2A	November 7, 2007

Name: ______________________________

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 7 pages (not counting this cover page). Please make sure that you have all 7 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
1	12	
2	20	
3	20	
4	20	
5	28	
TOTAL	100	

1. (12 points) Compute **A** if

$$\mathbf{AB} = \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

2. (20 points) Let

$$\mathbf{M} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix},$$

and suppose that $\det \mathbf{M} = 3$. Calculate the determinants of the following matrices. You will not receive any credit unless you include a one-sentence justification of your answer.

(a) $\mathbf{A} = \begin{bmatrix} 2\mathrm{a} & 2\mathrm{b} & 2\mathrm{c} \\ 2\mathrm{d} & 2\mathrm{e} & 2\mathrm{f} \\ 2\mathrm{g} & 2\mathrm{h} & 2\mathrm{i} \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2\mathrm{a} & 2\mathrm{b} & 2\mathrm{c} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \\ 3\mathrm{d} & 3\mathrm{e} & 3\mathrm{f} \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 3\mathrm{d}+2\mathrm{a} & 3\mathrm{e}+2\mathrm{b} & 3\mathrm{f}+2\mathrm{c} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{g} \\ \mathrm{b}+2\mathrm{a} & \mathrm{e}+2\mathrm{d} & \mathrm{h}+2\mathrm{g} \\ \mathrm{c} & \mathrm{f} & \mathrm{i} \end{bmatrix}$

3. (20 points) Let

$$\mathbf{A} = \frac{1}{9}\begin{bmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{bmatrix}$$

and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

(a) Find a basis for the range of L. What is the dimension of the range of L?

(b) Find a basis for the kernel of L. What is the dimension of the kernel of L?

4. (20 points) Recall that $\mathbb{P}_3$ is the vector space of all polynomials $p(t)$ of degree at most 3. Consider the subset S of $\mathbb{P}_3$ consisting of polynomials p such that $p(2) = 0$. In other words, the polynomial p is in S if 2 is a root of p.

   (a) Show that S is a vector subspace of $\mathbb{P}_3$.

Problem 4 (continued):

(b) Determine a basis for S. Justify that your answer is a basis. What is the dimension of S?

5. (28 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

   (a) The transpose of an elementary matrix is elementary.

   (b) If $\det\left(\mathbf{A}^3\right) = 0$, then $\mathbf{A}$ is not invertible.

Question 5 (continued):

(c) If H and K are subspaces of a vector space V, then their intersection $H \cap K$ is a subspace of V.

(d) If H and K are subspaces of a vector space V, then their union $H \cup K$ is a subspace of V.

MA 242	Exam 3A	December 12, 2007

Name: ______________________________

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 6 pages (not counting this cover page). Please make sure that you have all 6 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

1. (20 points) Diagonalize the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 7 & -15 \\ 1 & -1 \end{bmatrix}.$$

In other words, write $\mathbf{A}$ as $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix.

2. (20 points) Using projections, find the equation of the line that gives the best least squares fit for the four data points $(1, 0)$, $(2, 1)$, $(4, 2)$, and $(5, 3)$.

3. (20 points) Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Calculate the distance of $\mathbf{y}$ to W.

4. (20 points) Consider the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \quad \text{and the three vectors } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Show that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of $\mathbf{A}$.

(b) Compute an orthogonal diagonalization of $\mathbf{A}$.

5. (20 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

   (a) If $\mathbf{A}$ is a symmetric matrix, then $(\text{Col } \mathbf{A})^{\perp} = \text{Nul } \mathbf{A}$.

   (b) Let $W = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2\}$. If $\mathbf{v}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\mathbf{v} \in W^{\perp}$.

Question 5 (continued):

(c) Every orthogonal matrix is invertible.

(d) All eigenvectors of a square matrix $\mathbf{A}$ are elements of Col $\mathbf{A}$.

MA 242 Final A December 19, 2007

Name: ______________________________

Directions: Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 7 questions on 13 pages (not counting this cover page). Please make sure that you have all 13 pages of questions.

Do not write in the following box:

PROBLEM	POSSIBLE	SCORE
1	10	
2	14	
3	16	
4	8	
5	18	
6	10	
7	24	
TOTAL	100	

1. (10 points) Consider the subset H of $\mathbb{R}^4$ given by

$$\left\{ \begin{bmatrix} a - 2b - 2c + 3d \\ 2a - 4b - 3c + 9d \\ a - 2b - c + 6d \\ c + 3d \end{bmatrix} \; : \; a, b, c, d \text{ in } \mathbb{R} \right\}.$$

(a) Explain why H is a subspace of $\mathbb{R}^4$.

(b) Find a basis for H and calculate its dimension.

2. (14 points) Consider the system

$$\begin{aligned} x_1 + 3x_4 &= 4 \\ x_2 + x_3 - 3x_4 &= 2 \\ -x_1 - 3x_4 + x_5 &= -1 \\ 3x_1 + 9x_4 - 2x_5 &= 6 \end{aligned}$$

of four equations in five unknowns.

(a) Express its solution set in parametric vector form.

(Additional blank space on top of next page.)

Problem 2 (continued):

(b) Is it possible to change the constants on the right-hand side of the system so that the new system is inconsistent? In order to receive any credit, you must justify your answer.

3. (16 points) Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Calculate the characteristic polynomial and eigenvalues of $\mathbf{A}$.

Problem 3 (continued): Here is the matrix $\mathbf{A}$ again:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(b) Diagonalize $\mathbf{A}$. In other words, find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. (**You do not need to calculate $\mathbf{P}^{-1}$**.)

4. (8 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first performs a horizontal shear that transforms $\mathbf{e}_2$ to $\mathbf{e}_2 - 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

   (a) Find the standard matrix representation for T.

   (b) Show that T is invertible and find a formula for T^{-1}.

5. (18 points) Let

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 0 \\ 3 & 0 & 6 \end{bmatrix}.$$

(a) Compute $\mathbf{A}^{-1}$. You may use your calculator to double check your answer, but you will not get any credit unless you show enough work so that I can be sure that you can do this problem without your calculator.

Problem 5. (continued)

(b) Write $\mathbf{A}^{-1}$ as a product of elementary matrices. You do not need to multiply the elementary matrices together when you write $\mathbf{A}^{-1}$ as a product.

(c) Using the calculations that you have already made, determine the value of $\det \mathbf{A}$. At what point in those calculations were you sure about this value?

6. (10 points) The trace of a matrix is the sum of its entries along the diagonal. For example, the trace of the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $a + d$. Consider the subset S of the vector space $M_{2\times 2}$ of all 2×2 matrices that consists of all matrices whose trace is zero.

(a) Show that S is a vector subspace of $M_{2\times 2}$.

Problem 6 (continued):

(b) Determine a basis for S. What is the dimension of S? Justify that your answer is a basis.

7. (24 points) Are the following statements true or false? You must justify your answers to receive any credit.

   (a) A homogeneous linear system of three equations in five variables has at least two free variables.

   (b) The columns of any 3×4 matrix are linearly dependent.

   (c) Let T be a linear transformation. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent.

Problem 7 (continued):

(d) Let $\mathbf{A}$ be an $m \times n$ matrix. The subspace Col $\mathbf{A}$ is the set of all vectors of the form $\mathbf{Ax}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(e) If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

(f) A square matrix $\mathbf{A}$ is not invertible if and only if 0 is an eigenvalue of $\mathbf{A}$.

Problem 7 (continued):

(g) If the square matrix $\mathbf{A}$ is diagonalizable, then the columns of $\mathbf{A}$ are linearly independent.

(h) If a matrix $\mathbf{U}$ has orthonormal columns, then $\mathbf{U}\mathbf{U}^T = \mathbf{I}$.