**MA 242 Exams**
**Fall 2008**

Here are the four exams that I gave in my MA 242 class in the Fall of 2008. You should use them to get an idea of the format of a typical test and to see the types of questions I ask. *You should not assume that the test questions this semester will be on the same topics.* In fact, you are always responsible for *all* of the material that we cover in class as well as *all* of the designated material from your text. The best way to study for my exams is to be sure that you are very comfortable with the homework assignments and the examples that I present in class. My tests often vary in difficulty (as you can see here), and your grade for the examination will be determined by a curve that will be announced in class after the examination is graded.

MA 242 Exam 1A October 1, 2008

Name: ____________________

**Directions:** Please do all of your work in this exam booklet and make sure that you cross out any work that we should ignore when we grade. **Books and extra papers are not permitted.** If you have a question about a problem, please ask. Remember: answers that are written logically and clearly will receive higher scores. There are 5 questions on 6 pages (not counting this cover page). Please make sure that you have all 6 pages of questions.

Do not write in the following box:

| PROBLEM | POSSIBLE | SCORE |
|---|---|---|
| 1 | 15 | |
| 2 | 16 | |
| 3 | 21 | |
| 4 | 20 | |
| 5 | 28 | |
| TOTAL | 100 | |

1. (15 points) Row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & -2 & 0 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$$

to **reduced row echelon form** (RREF). Do only one row operation at a time and specify that operation when you perform it. Indicate when you first arrive at a matrix in **echelon form** (REF). What are the pivot positions of $\mathbf{A}$?

2. (16 points) Find the value(s) of $h$ such that the following set of vectors is linearly independent:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix} \right\}$$

3. (21 points) Which of the following functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ are linear? If the function is linear, what is its standard matrix representation? If the function is not linear, justify your answer by giving an example of one of the linearity properties that does not hold.

   (a) $T(x_1, x_2) = (3x_1 - x_2, 2|x_1|)$

   (b) $T(x_1, x_2) = (x_2 - x_1, \sin x_2)$

   (c) $T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2)$

4. (20 points) Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -7 & 5 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 6 & 8 \end{bmatrix}$$

let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Find all vectors $\mathbf{x}$ in $\mathbb{R}^4$ such that

$$T(\mathbf{x}) = \begin{bmatrix} 8 \\ 1 \\ 1 \\ 9 \end{bmatrix}.$$

5. (28 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

   (a) Each matrix is row equivalent to a unique matrix in echelon form.

   (b) An $m \times n$ matrix $\mathbf{A}$ defines a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ by the formula $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

Question 5 (continued):

(c) The columns of any $4 \times 5$ matrix are linearly dependent.

(d) Suppose that $\mathbf{A}$ is an $m \times n$ matrix with $m \geq 2$ and $\mathbf{B}$ is an $n \times p$ matrix. Then the second row of $\mathbf{AB}$ is the second row of $\mathbf{A}$ multiplied on the right by $\mathbf{B}$.

| MA 242 | Exam 2A | October 29, 2008 |
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Name: ______________________________

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Do not write in the following box:

| PROBLEM | POSSIBLE | SCORE |
|---|---|---|
| 1 | 18 | |
| 2 | 18 | |
| 3 | 16 | |
| 4 | 20 | |
| 5 | 28 | |
| TOTAL | 100 | |

1. (18 points) Compute the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -4 & -1 & 0 \end{bmatrix}$$

using cofactor expansion until you arrive at $2 \times 2$ matrices. You may *check* your answer using your calculator, but you will not receive any credit unless you show all steps in the computation.

2. (18 points) Use Cramer's Rule to compute the solution to the system

$$\begin{aligned} 2x_1 + x_2 &= 7 \\ -3x_1 + x_3 &= -8 \\ x_2 + 2x_3 &= -3. \end{aligned}$$

3. (16 points) Find the matrix $\mathbf{A}$ whose inverse is $\mathbf{A}^{-1} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$.

4. (20 points) Recall that $\mathbb{P}_n$ is the vector space of all polynomials $p(t)$ of degree at most $n$. Let $L : \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation given by

$$L(p(t)) = (t + 2)\, p(t).$$

(a) Show that $L$ is a linear transformation.

Question 4 (continued):

(b) What are the kernel and the range of $L$? Describe the range of $L$ as a single equation for the coefficients $a_3$, $a_2$, $a_1$, and $a_0$ of $p(t) = a_3t^3 + a_2t^2 + a_1t + a_0$. This equation should not include the variable $t$. Provide a brief justification for both of your answers.

5. (28 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

   (a) Every elementary matrix is invertible.

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent square matrices, then $\det \mathbf{A} = \det \mathbf{B}$.

Question 5 (continued):

(c) Row operations on a matrix can change the null space.

(d) If $\mathbf{A}$ is a square matrix and the equation $\mathbf{A}\mathbf{x} = \mathbf{e}_1$ has a unique solution, then $\mathbf{A}$ is invertible.

| MA 242 | Exam 3A | December 10, 2008 |
|---|---|---|

Name: ______________________________________________

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Do not write in the following box:

| PROBLEM | POSSIBLE | SCORE |
|---|---|---|
| 1 | 18 | |
| 2 | 18 | |
| 3 | 18 | |
| 4 | 18 | |
| 5 | 28 | |
| TOTAL | 100 | |

1. (18 points) Consider the subspace $H$ of $\mathbb{R}^4$ given by

$$\left\{ \begin{bmatrix} a+2b-2c+d \\ 2a+4b-3c+6d \\ -3a-6b+7c+d \\ c+4d \end{bmatrix} \ : \ a,b,c,d \text{ in } \mathbb{R} \right\}.$$

Find a basis for $H$. What is the dimension of $H$?

2. (18 points) Let

$$\mathbf{A} = \begin{bmatrix} -4 & -2 & 2 & 1 \\ -2 & -1 & -3 & -4 \\ 3 & -6 & 2 & 1 \\ 0 & 0 & 1 & -7 \end{bmatrix}.$$

Compute a basis for the $\lambda = -5$ eigenspace of $\mathbf{A}$.

3. (18 points) Find the point on the plane $x_1 + 2x_2 - x_3 = 0$ that is closest to the point $(2, 4, 3)$.

4. (18 points) Determine the $3 \times 3$ projection matrix $\mathbf{P}$ that corresponds to orthogonal projection of $\mathbb{R}^3$ onto the line that is spanned by the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

What are the eigenvalues of $\mathbf{P}$? Explain why you know the answer to this question without doing any calculation. Also, in one or two sentences, describe the corresponding eigenspaces.

5. (28 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

   (a) If there exists a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space $V$, then $\dim V \geq p$.

   (b) Let $\mathbf{A}$ be an $m \times n$ matrix. Then $\dim \text{Row } \mathbf{A} + \dim \text{Nul } \mathbf{A} = n$.

Question 5 (continued):

(c) If 0 is an eigenvalue for the $n \times n$ matrix $\mathbf{A}$, then the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^n$.

(d) Let $\mathbf{A}$ be an $m \times n$ matrix. Every vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{p} + \mathbf{u}$ where $\mathbf{p}$ is in Row $\mathbf{A}$ and $\mathbf{u}$ is in Nul $\mathbf{A}$.

| MA 242 | Final A | December 20, 2008 |
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Name: ____________________

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Do not write in the following box:

| PROBLEM | POSSIBLE | SCORE |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 14 | |
| 4 | 14 | |
| 5 | 14 | |
| 6 | 14 | |
| 7 | 24 | |
| TOTAL | 100 | |

1. (10 points) Let

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) Write $\mathbf{v}$ as the sum of a vector in the line spanned by $\mathbf{w}$ and a vector orthogonal to $\mathbf{w}$.

   (b) Compute the distance of $\mathbf{v}$ to the line spanned by $\mathbf{w}$.

2. (10 points) Let $\mathbf{A}$ and $\mathbf{B}$ be $4 \times 4$ matrices with $\det \mathbf{A} = 2$ and $\det \mathbf{B} = -3$. Compute:

(a) $\det 3\mathbf{A}$

(b) $\det \mathbf{B}^3$

(c) $\det \mathbf{AB}$

(d) $\det \mathbf{A}^{\mathrm{T}}\mathbf{A}$

(e) $\det \mathbf{B}^{-1}\mathbf{AB}$

3. (14 points) Consider the set $S$ of all vectors

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

in $\mathbb{R}^4$ such that

$$\begin{aligned} a - 2b + 2c + d &= 0 \\ -3a + 6b - 5c - d &= 0 \\ 4a - 8b + 9c + 6d &= 0. \end{aligned}$$

(a) Why is $S$ a subspace of $\mathbb{R}^4$?

(b) Determine the dimension of $S$ and find a basis.

4. (14 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

(a) Without doing any computation, explain why $\lambda = 5$ is an eigenvalue.

(b) What's the "easy" way to show that $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is an eigenvector?

(c) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. **You do not need to calculate $\mathbf{P}^{-1}$.**

5. (14 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first rotates the plane by 45° in the counterclockwise direction, then dilates the plane by a factor of 2, and finally reflects the plane in the $x_1$-axis.

   (a) Find the standard matrix representation for $T$.

   (b) Let $P$ be the parallelogram determined by the two vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   Calculate the area of $T(P)$.

6. (14 points) **Note that there is a second part to this problem on the next page.** Recall that a matrix is upper triangular if all of its entries below the diagonal are zero. For example, an upper-triangular $3 \times 3$ matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{bmatrix}$$

where the entries $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,2}$, $a_{2,3}$, and $a_{3,3}$ can be any real numbers.

(a) Show that the subset $S$ of all upper-triangular matrices in $M_{3\times 3}$ is a vector subspace of $M_{3\times 3}$.

Problem 6 (continued):

(b) Specify a basis for $S$ and show that it is a basis. What is the dimension of $S$?

7. (24 points) Are the following statements true or false? You must justify your answers to receive any credit.

   (a) Row operations on a matrix **A** can change the linear dependence relations among the rows of **A**.

   (b) A square matrix is invertible if and only if it is the product of elementary matrices.

Problem 7 (continued):

(c) A basis is a spanning set that is as large as possible.

(d) If $\lambda$ is an eigenvalue for the $n \times n$ matrix $\mathbf{A}$ and $\mu$ is an eigenvalue for the $n \times n$ matrix $\mathbf{B}$, then the product $\lambda\mu$ is an eigenvalue for the matrix $\mathbf{AB}$.

Problem 7 (continued):

(e) Every projection matrix is orthogonal.

(f) Every projection matrix is diagonalizable.