Projection Matrices

We discussed projection matrices briefly when we discussed orthogonal projection. In particular, we discussed the following theorem.

Theorem. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$\mathbf{U} = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k \end{array} \right].$$

Then $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T\mathbf{v}$.

The matrix $\mathbf{U}\mathbf{U}^T$ is called the *projection matrix* for the subspace W. It does not depend on the choice of orthonormal basis.

What if we do not start with an orthonormal basis of W?

Theorem. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be any basis for a subspace W of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_k \end{array} \right].$$

Then the projection matrix for W is $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.

To see why this formula is true, we need a lemma.

Lemma. Suppose $\mathbf{A}$ is an $n \times k$ matrix whose columns are linearly independent. Then $\mathbf{A}^T\mathbf{A}$ is invertible.

To see why this lemma is true, consider the transformation $\mathbf{A} : \mathbb{R}^k \to \mathbb{R}^n$ determined by $\mathbf{A}$. Since the columns of $\mathbf{A}$ are linearly independent, this transformation is one-to-one. Moreover, the null space of $\mathbf{A}^T$ is orthogonal to the column space of $\mathbf{A}$. Consequently, $\mathbf{A}^T$ is one-to-one on the column space of $\mathbf{A}$, and as a result, $\mathbf{A}^T\mathbf{A} : \mathbb{R}^k \to \mathbb{R}^k$ is one-to-one. By the Invertible Matrix Theorem, $\mathbf{A}^T\mathbf{A}$ is invertible.

Now we can compute the projection matrix for the column space of $\mathbf{A}$. (Note that $W = \text{Col } \mathbf{A}$.) Any element of the column space of the matrix $\mathbf{A}$ is a linear combination of the columns of $\mathbf{A}$, that is,

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_k\mathbf{a}_k.$$

If we let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix},$$

then

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_k\mathbf{a}_k = \mathbf{A}\mathbf{x}.$$

Given $\mathbf{v}$ in $\mathbb{R}^n$, we denote by $\mathbf{x}_p$ the $\mathbf{x}$ that corresponds to the projection of $\mathbf{v}$ onto W. In other words, let

$$\text{proj}_W\mathbf{v} = \mathbf{A}\mathbf{x}_p.$$

We find the projection matrix by calculating $\mathbf{x}_p$.

The projection of $\mathbf{v}$ onto W is characterized by the fact that

$$\mathbf{v} - \text{proj}_W\mathbf{v}$$

is orthogonal to each vector $\mathbf{w}$ in W, that is,

$$\mathbf{w}\boldsymbol{\cdot}(\mathbf{v} - \text{proj}_W\mathbf{v}) = 0$$

for all $\mathbf{w}$ in W. Since $\mathbf{w} = \mathbf{A}\mathbf{x}$ for some $\mathbf{x}$, we have

$$\mathbf{A}\mathbf{x}\boldsymbol{\cdot}(\mathbf{v} - \mathbf{A}\mathbf{x}_p) = 0$$

for all $\mathbf{x}$ in $\mathbb{R}^k$. Writing this dot product in terms of matrices yields

$$(\mathbf{A}\mathbf{x})^T(\mathbf{v} - \mathbf{A}\mathbf{x}_p) = 0,$$

which is equivalent to

$$(\mathbf{x}^T\mathbf{A}^T)(\mathbf{v} - \mathbf{A}\mathbf{x}_p) = 0.$$

Converting back to dot products, we have

$$\mathbf{x}\boldsymbol{\cdot}\mathbf{A}^T(\mathbf{v} - \mathbf{A}\mathbf{x}_p) = 0.$$

In other words, the vector $\mathbf{A}^T(\mathbf{v} - \mathbf{A}\mathbf{x}_p)$ is orthogonal to every vector $\mathbf{x}$ in $\mathbb{R}^k$. The only vector in $\mathbb{R}^k$ with this property is the zero vector, so we may conclude that

$$\mathbf{A}^T(\mathbf{v} - \mathbf{A}\mathbf{x}_p) = \mathbf{0}.$$

We get

$$\mathbf{A}^T\mathbf{v} = \mathbf{A}^T\mathbf{A}\mathbf{x}_p.$$

From the lemma, we know that $\mathbf{A}^T\mathbf{A}$ is invertible, and we have

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{v} = \mathbf{x}_p.$$

Since $\mathbf{A}\mathbf{x}_p$ is the desired projection, we have

$$\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{v} = \text{proj}_W\mathbf{v}.$$

We conclude that the projection matrix for W is

$$\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T.$$

Note that any projection matrix $\mathbf{P}$ satisfies the two properties

1. $\mathbf{P}^2 = \mathbf{P}$, and
2. $\mathbf{P}$ is symmetric.

It is also true that any matrix that satisfies these two properties is the projection matrix for some subspace of $\mathbb{R}^n$ (see Exercise 36 in Section 7.1 of Lay).