The Casting-Out Procedure

Given a vector subspace S spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can obtain a basis B for S by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
   (a) (cast out) $B_i = B_{i-1}$ if $\mathbf{v}_i$ is in $\operatorname{Span} B_{i-1}$, or
   (b) (keep) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if $\mathbf{v}_i$ is not in $\operatorname{Span} B_{i-1}$.

Then the final result B_k is a basis for S.

To prove this theorem, we must show that the casting-out procedure produces a linearly independent set that still spans S.

Linear independence: Let B_i be the first step in the procedure for which B_i is linearly dependent. Then $\mathbf{v}_i$ is an element of B_i, but it is also a linear combination of vectors in B_{i-1}. This situation contradicts part 2 of the procedure. Consequently, the sets B_i are linearly independent for $i = 1, \ldots, k$.

Spanning: We must show that $\operatorname{Span} B_k = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. To do so, we prove that

$$\operatorname{Span} B_i = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$$

for $i = 1, \ldots, k$ by induction on i.

Certainly $\operatorname{Span} B_1 = \operatorname{Span}\{\mathbf{v}_1\}$, so we assume that $\operatorname{Span} B_{i-1} = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$ and show that $\operatorname{Span} B_i = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$. If $B_i = B_{i-1}$, then $\mathbf{v}_i$ is a linear combination of the vectors in B_{i-1}, and therefore,

$$\begin{aligned} \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\} &= \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\} \\ &= \operatorname{Span} B_{i-1} \\ &= \operatorname{Span} B_i. \end{aligned}$$

If $B_i \neq B_{i-1}$, then every vector $\mathbf{v}$ in $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ can be written as

$$\mathbf{v} = \mathbf{w} + r_i \mathbf{v}_i,$$

where $\mathbf{w}$ is in $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$. By the inductive hypothesis, $\mathbf{w}$ is in $\operatorname{Span} B_{i-1}$, and therefore, $\mathbf{v}$ is in $\operatorname{Span} B_i$.