MA 713

Derivatives of the Integral Representation

Theorem. *Let γ be a simple, closed, positively-oriented contour in $\mathbb{C}$. If $g(z) : \gamma \to \mathbb{C}$ is continuous on γ, then the function*

$$G(z) = \int_\gamma \frac{g(w)}{w - z}\, dw$$

is analytic and infinitely differentiable on the domain U enclosed by γ. In fact,

$$G^{(k)}(z) = k! \int_\gamma \frac{g(w)}{(w - z)^{k+1}}\, dw.$$

Outline of Proof. Let

$$G_n(z) = \int_\gamma \frac{g(w)}{(w - z)^n}\, dw.$$

We use induction on n to show that $G_n(z)$ is differentiable on U, and

$$G_n'(z) = nG_{n+1}(z).$$

The $n = 1$ case: Note that

$$G_1(z_2) - G_1(z_1) = \int_\gamma \frac{(z_2 - z_1)g(w)}{(w - z_1)(w - z_2)}\, dw$$

because

$$\frac{1}{w - z_2} - \frac{1}{w - z_1} = \frac{z_2 - z_1}{(w - z_1)(w - z_2)}.$$

(A) The continuity of G_1 follows:

(B) The differentiability of $G_1(z)$ follows from an application of (A):

The inductive step: We assume that the proposition is true for $n-1$ and prove it for n.

Observe that

$$G_n(z_2) - G_n(z_1) = \left[\int_\gamma \frac{g(w)}{(w-z_2)^{n-1}(w-z_1)}\, dw - \int_\gamma \frac{g(w)}{(w-z_1)^n}\, dw \right] + (z_2 - z_1) \int_\gamma \frac{g(w)}{(w-z_2)^n (w-z_1)}\, dw.$$

because the middle term is $G_n(z_1)$ and

$$\frac{1}{(w-z_2)^n} = \frac{1}{(w-z_2)^{n-1}(w-z_1)} + \frac{z_2 - z_1}{(w-z_2)^n (w-z_1)}.$$

(A) Continuity of G_n:

(B) Differentiability of G_n: