MA 713

Five Versions of Cauchy's Theorem

Terminology: When we say that a function f is analytic on a closed set C, we mean that it is analytic on some open set U that contains C.

In what follows, R is a rectangle whose sides are parallel to the real and imaginary axes, and $\mathbb{D}_r(z_0)$ is the open disk of radius r centered at z_0 in $\mathbb{C}$.

Five versions of Cauchy's Theorem:

1. Green's Theorem version: Let γ be a simple, closed curve in $\mathbb{C}$ and D be the domain enclosed by γ. If f is analytic on $\gamma \cup D$ and f' is continuous on $\gamma \cup D$, then

$$\int_\gamma f(z)\, dz = 0.$$

2. Goursat's version: If f is analytic on the rectangle R, then

$$\int_{\partial R} f(z)\, dz = 0.$$

3. Let R' be a subset of the rectangle R obtained by omitting a finite number of interior points a_j of R. If f is analytic on R' and

$$\lim_{z \to a_j} (z - a_j) f(z) = 0$$

for all a_j, then

$$\int_{\partial R} f(z)\, dz = 0.$$

4. If the function f is analytic on the disk $\mathbb{D}_r(z_0)$, then the differential $f(z)\, dz$ is exact on $\mathbb{D}_r(z_0)$. In other words, f satisfies the Path Independence Theorem on $\mathbb{D}_r(z_0)$.

5. Let $\mathbb{D}'_r(z_0)$ be a subset of the disk $\mathbb{D}_r(z_0)$ obtained by omitting a finite number of interior points a_j of $\mathbb{D}_r(z_0)$. If f is analytic on $\mathbb{D}'_r(z_0)$ and

$$\lim_{z \to a_j} (z - a_j) f(z) = 0$$

for all a_j, then the differential $f(z)\, dz$ is exact on the punctured disk $\mathbb{D}'_r(z_0)$.