

MA 713  
Five Versions of Cauchy's Theorem

**Terminology:** When we say that a function  $f$  is analytic on a closed set  $C$ , we mean that it is analytic on some open set  $U$  that contains  $C$ .

Let  $R$  be a rectangle whose sides are parallel to the real and imaginary axes, and let  $\mathbb{D}_r(z_0)$  be any open disk in  $\mathbb{C}$ .

**Five versions of Cauchy's Theorem:**

1. Green's Theorem version: Let  $\gamma$  be a simple, closed curve in  $\mathbb{C}$  and  $D$  be the domain enclosed by  $\gamma$ . If  $f$  is analytic on  $\gamma \cup D$  and  $f'$  is continuous on  $\gamma \cup D$ , then

$$\int_{\gamma} f(z) dz = 0.$$

2. Goursat's version: If  $f$  is analytic on  $R$ , then

$$\int_{\partial R} f(z) dz = 0.$$

3. Let  $R'$  be a subset of  $R$  obtained by omitting a finite number of interior points  $a_i$  of  $R$ . If  $f$  is analytic on  $R'$  and

$$\lim_{z \rightarrow a_i} (z - a_i)f(z) = 0$$

for all  $a_i$ , then

$$\int_{\partial R} f(z) dz = 0.$$

4. If  $f$  is analytic on  $\mathbb{D}_r(z_0)$ , then  $f(z) dz$  is exact on  $\mathbb{D}_r(z_0)$ . In other words,  $f$  satisfies the Path Independence Theorem on  $\mathbb{D}_r(z_0)$ .

5. Let  $\mathbb{D}'_r(z_0)$  be a subset of  $\mathbb{D}_r(z_0)$  obtained by omitting a finite number of interior points  $a_i$  of  $\mathbb{D}_r(z_0)$ . If  $f$  is analytic on  $\mathbb{D}'_r(z_0)$  and

$$\lim_{z \rightarrow a_i} (z - a_i)f(z) = 0$$

for all  $a_i$ , then  $f(z) dz$  is exact on  $\mathbb{D}'_r(z_0)$ .