MA 713 March 18, 2005

Take Home Examination I

**Directions:** Here are the rules that you must follow while taking this examination:

1. You must submit your solutions to this examination at **noon on Friday, March 25**.
2. You cannot talk to or in any way consult with any human regarding **any** aspect of this examination.
3. The only resources of any kind that you may consult during the examination are your textbook (Ahlfors) and your class notes.

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1. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z} & \text{for } z \neq 0; \\ 0 & \text{for } z = 0. \end{cases}$$

   Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but that the derivative of $f$ fails to exist there.

2. (Ahlfors, p. 130, #2) Show that the only entire functions that have a nonessential singularity at $\infty$ are polynomials.

3. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \mathrm{Re}[f(z)]$ is bounded above on $\mathbb{R}^2$, i.e., there exist an $M$ such that $u(x, y) \leq M$ for all $(x, y) \in \mathbb{R}^2$. Show that $u(x, y)$ is constant on $\mathbb{R}^2$.

4. (Ahlfors, p. 130, #4) Show that any function that is meromorphic in the extended plane $\overline{\mathbb{C}}$ is rational.

5. (Ahlfors, p. 136, #5) Use Schwarz's lemma to prove that every one-to-one conformal mapping of a disk onto another disk (or a half plane) is given by a linear fractional transformation.

6. Let $g$ be a continuous complex-valued function defined on the unit circle $|z| = 1$. Define

$$f(z) = \begin{cases} g(z) & \text{for } |z| = 1; \\ \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{g(w)}{w - z}\, dw & \text{for } |z| < 1, \end{cases}$$

   where $\gamma$ is the positively-oriented unit circle. Is $f(z)$ continuous on the closed unit disk $|z| \leq 1$?