MA 713 May 4, 2005

Take Home Examination II

Directions: Here are the rules that you must follow while taking this examination:

1. You must submit your solutions to this examination by **noon on Wednesday, May 11** at my office (MCS 255).
2. You cannot talk to or in any way consult with any human regarding **any** aspect of this examination.
3. The only resources of any kind that you may consult during the examination are your textbook (Ahlfors) and your class notes.

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1. (Ahlfors, p. 184, #5) The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$. Show that the Fibonacci numbers are Taylor coefficients of a rational function, and determine a closed form expression for c_n.
2. (Ahlfors, p. 186, #4) Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form
$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$
where the numbers B_k are known as the Bernoulli numbers. Calculate B_1, B_2, B_3. (By Sec. 2.1, Ex. 5, the B_k are all positive.)
3. Let $\overline{\mathbb{D}_r} = \{z \,|\, |z| \leq r\}$. What is the Laurent expansion of a univalent analytic function $f : \mathbb{C} - \overline{\mathbb{D}_r} \to \mathbb{C}$ such that $f(\infty) = \infty$?
4. (Ahlfors, p. 190, #2) Express
$$\sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3}$$
in closed form.
5. Show that
$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{7}\right)\cdots$$
6. Let U be an open subset of the complex plane and let $\mathcal{F}$ be a family of analytic functions defined on a domain V. Prove that, if every $f \in \mathcal{F}$ "omits" U (i.e., $f(V) \cap U = \emptyset$), then $\mathcal{F}$ is a normal family.
7. Let
$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$
Show that $f(z)$ cannot be analytically continued to any connected, open set that properly contains the open unit disk centered at the origin.