MA 771 Exercises

2.1. Suppose p and $q \neq 0$ are relatively prime integers and that k is an integer multiple of q. Let

$$F(x) = x + \epsilon \sin(2k\pi x),$$

where $0 < \epsilon < 1/(2k\pi)$. The map $F : \mathbb{R} \to \mathbb{R}$ induces a map $f : S^1 \to S^1$ using the universal covering map $u : \mathbb{R} \to S^1$. Let $g = R_{p/q} \circ f$.

(a) What role does p play in the dynamics of g?

(b) Construct a lift $G : \mathbb{R} \to \mathbb{R}$ of g.

(c) Using G, calculate the rotation number of g.

2.2. Given a lift $F : \mathbb{R} \to \mathbb{R}$ of an orientation preserving homeomorphism $f : S^1 \to S^1$, let ρ_F denote the rotation number of F.

(a) Prove that, if $\rho_F = 0$, then $F(x) \geq x$ for some $x \in \mathbb{R}$. (This completes the $\rho_F = 0$ case discussed in class.)

(b) Prove that $\rho_{(F^m+k)} = m(\rho_F) + k$.

(c) Prove that, if ρ_F is rational, then f has a periodic point.

(d) Let $u : \mathbb{R} \to S^1$ be the universal covering map. Prove that the number $u(\rho_F)$ is independent of the choice of the lift F of f.

2.3. Robinson 2.30 (p. 63)

2.4. Give a different looking proof of the existence of the rotation number as follows:

(a) Given $x \in \mathbb{R}$, let $a_n = F^n(x) - x$ and k_n be the greatest integer in a_n. Then $x + k_n \leq F^n(x) < x + k_n + 1$. Show that a_n is a subadditive sequence (i.e., $a_{m+n} \leq a_m + a_n + c$ for some constant c) by considering

$$\begin{aligned} a_{m+n} = &\Big[F^m(x+k_n) - (x+k_n)\Big] + \Big[F^n(x) - x\Big] + \\ &\Big[\Big(F^{m+n}(x) - F^m(x+k_n)\Big) - \Big(F^n(x) - (x+k_n)\Big)\Big]. \end{aligned}$$

(b) Let μ be the minimum value of $F - Id$ on the interval $[0,1]$. Show that $(a_n/n) \geq \mu$ for all n by considering

$$\frac{a_n}{n} = \frac{1}{n}\sum_{i=0}^{n-1}(F^{i+1}(x) - F^i(x)).$$

(c) Why does the rotation number exist?