MA 771 **February 13, 2004**

Equivalence of Norms on $\mathbb{R}^n$

Recall that a norm on $\mathbb{R}^n$ is a function $N : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following three properties:

1. $N(x) \geq 0$ for all x, and $N(x) = 0$ if and only if $x = 0$.
2. $N(x + y) \leq N(x) + N(y)$.
3. $N(rx) = |r|N(x)$ for all $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

For example, the usual length function $|x|$ is a norm, and the max norm, as defined by

$$|x|_{\max} = \max\{|x_1|, |x_2|, \ldots, |x_n|\},$$

is a norm. (You should verify the above three properties for the max norm.)

The goal of this handout is to show that any two norms on $\mathbb{R}^n$ induce the same topology on $\mathbb{R}^n$. In other words, convergence in one norm is the same as convergence in any other norm (on $\mathbb{R}^n$).

Definition. A norm $N(x)$ is equivalent to the standard norm on $\mathbb{R}^n$ if there exist positive constants A and B such that

$$A|x| \leq N(x) \leq B|x|$$

for all $x \in \mathbb{R}^n$.

Step 1. Show that $|x|_{\max}$ is equivalent to the standard norm by proving that

$$\frac{1}{\sqrt{n}}|x| \leq |x|_{\max} \leq |x|.$$

Step 2. Show that any norm $N : \mathbb{R}^n \to \mathbb{R}$ is a continuous function on $\mathbb{R}^n$. In fact, let

$$M = \max\{N(e_1), N(e_2), \ldots, N(e_n)\}$$

where e_i represents the i-th standard basis vector in $\mathbb{R}^n$. Show that $N(x) \leq Mn|x|$ for all $x \in \mathbb{R}^n$ and then show that $N(x)$ is continuous.

Step 3. Let A be the minimum value and B be the maximum value of $N(x)$ on the compact set $\{x \in \mathbb{R}^n \mid |x| = 1\}$. Show that $N(x)$ is equivalent to $|x|$ using these choices of A and B.