Problem 1 (15 points).
Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of drug that is present at the time, i.e.,

$$\frac{dC}{dt} = -kC \quad \text{with } k > 0 \text{ constant}.$$  

(a) If $C_0$ is the concentration given at time $t = 0$, find the concentration at time $t$.

(b) Find the value of $k$ if the initial concentration is $C_0 = 50$ and if $C = 25$ at time $t = 30$.

Solution.

(a) We know that the solution to the initial value problem $\frac{dC}{dt} = -kC$, $C(0) = C_0$ is given by $C(t) = C_0 \cdot e^{-kt}$. (This is the “law of natural decay”, and follows from the fact that the differential equation is separable:

$$\frac{dC}{C} = -k \, dt$$  

$$\int \frac{dC}{C} = - \int k \, dt$$  

$$\ln |C| = -kt + A$$  

$$|C| = e^{-kt+A} = e^A \cdot e^{-kt}$$  

$$C(t) = B \cdot e^{-kt} \quad \text{(with } B = \pm e^A).$$

To fix $B$, note that $C(0) = C_0$ by assumption; therefore, $C_0 = B \cdot e^{-k0} = B$.

(b) For $C_0 = 50$, we find with (a) that the solution is given by $C(t) = 50 \cdot e^{-kt}$. To fix $k$, we have to use the fact that $C(30) = 25$, i.e., that the solution passes through the point $(t, C) = (30, 25)$. Hence,

$$25 = 50 \cdot e^{-k30}$$  

$$\frac{1}{2} = e^{-k30}$$  

$$\ln \frac{1}{2} = -k30$$  

$$k = -\frac{1}{30} \ln \frac{1}{2} = \frac{1}{30} \ln 2 \approx 0.0231.$$  

Problem 2 (15 points).
Find the solution of the differential equation

$$\frac{dx}{dt} = (1 - t) \cdot (1 + x)$$

that passes through the point $(t, x) = (0, 0)$.
Solution. The differential equation is separable, since the right-hand side is the product of a function of $t$ (the independent variable) times a function of $x$ (the dependent variable). Hence,
\[
\frac{dx}{1+x} = (1-t) \, dt
\]
\[
\int \frac{dx}{1+x} = \int (1-t) \, dt
\]
\[
\ln |1+x| = t - \frac{t^2}{2} + C
\]
\[
|1+x| = e^{t-\frac{t^2}{2}+C} = e^C \cdot e^{t-\frac{t^2}{2}}
\]
\[
1+x = A \cdot e^{t-\frac{t^2}{2}} \quad \text{(with } A = \pm e^C \text{)}
\]
\[
x(t) = A \cdot e^{t-\frac{t^2}{2}} - 1.
\]
This is the general solution of the equation. To find the solution which passes through the point $(0, 0)$, we have to fix the constant $A$:
\[
x(0) = 0 = A \cdot e^{0-\frac{0^2}{2}} - 1 = A - 1.
\]
Therefore, $A = 1$, and the unique solution for which $x(0) = 0$ is given by $x(t) = e^{t-\frac{t^2}{2}} - 1$.

Problem 3 (20 points).

Let
\[
f(x) = \begin{cases} 
\frac{\pi}{20} \sin \left(\frac{\pi x}{10}\right) & \text{if } 0 \leq x \leq 10, \\
0 & \text{if } x < 0 \text{ or } x > 10.
\end{cases}
\]

(a) Show that $f(x)$ is a probability density function.

(b) Find $P(X < 4)$.

Solution.

(a) To show that $f(x)$ is a probability density function, we have to show
\[
f(x) \geq 0 \quad \text{for all } x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1.
\]
Since $f(x) = 0$ for $x < 0$ or $x > 10$, we only have to consider $x \in [0, 10]$. To show that $f(x) \geq 0$ there, note that for $0 \leq x \leq 10$, there holds $0 \leq \frac{\pi x}{10} \leq \pi$. Since the sine function is positive on the interval $(0, \pi)$ and zero at $0$ and at $\pi$ and since $\frac{\pi}{20} > 0$, it follows that $f(x) \geq 0$ for $0 \leq x \leq 10$. To show that the second requirement holds, we compute
\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{10} \frac{\pi}{20} \sin \left(\frac{\pi x}{10}\right) \, dx = \frac{\pi}{20} \int_{0}^{\pi} \frac{10}{\pi} \sin (u) \, du = \frac{1}{2} (-\cos (u)) \bigg|_{0}^{\pi} = 1.
\]
(Here, we have made the substitution $u = \frac{\pi}{10} x$, $du = \frac{\pi}{10} dx$ to evaluate the integral.) Therefore, $f(x)$ is a probability density function.

(b) Since $f(x) = 0$ for $x < 0$, we have
\[
P(X < 4) = \int_{0}^{4} \frac{\pi}{20} \sin \left(\frac{\pi x}{10}\right) \, dx = \frac{1}{2} (-\cos (u)) \bigg|_{0}^{\frac{2\pi}{5}} \approx 0.3455.
\]
Hence, the probability that $X$ is less than 4 is about 34.55%.
Problem 4 (15 points).
Let the curve $C$ be defined by
\[ y(x) = \int_1^x \sqrt{t^2 - 1} \, dt \quad \text{for } 1 \leq x \leq 16. \]

Find the length $L$ of $C$. \textit{(Hint: Apply the Fundamental Theorem of Calculus to find $\frac{dy}{dx}$.)}

\textbf{Solution.} The curve $C$ is parametrized by $x$, with $x \in [1, 16]$. Hence, to find the length of $C$, we make use of the formula
\[ L = \int_1^{16} \sqrt{\left( \frac{dy}{dx} \right)^2 + 1} \, dx. \]

To compute $\frac{dy}{dx}$, we apply the Fundamental Theorem of Calculus:
\[ \frac{dy}{dx} = \frac{d}{dx} \left( \int_1^x \sqrt{t^2 - 1} \, dt \right) = \sqrt{x^2 - 1}. \]

Therefore, $\left( \frac{dy}{dx} \right)^2 = x^2 - 1$, and
\[ L = \int_1^{16} \sqrt{x^2 - 1 + 1} \, dx = \int_1^{16} x \, dx = \frac{4}{5} \cdot x^\frac{5}{2} \bigg|_1^{16} = \frac{124}{15} = 24.8. \]

Problem 5 (20 points).
Let the sequence $\{a_n\}$ be defined by $a_n = \frac{2n - 3}{3n + 4}$.

(a) Determine whether the sequence is increasing, decreasing, or not monotonic.

(b) Determine whether the sequence is bounded.

\textbf{Solution.}

(a) Computing the first four terms of the sequence, we find $a_1 = \frac{-1}{7} \approx -0.1429$, $a_2 = \frac{1}{10} = 0.1$, $a_3 = \frac{3}{13} \approx 0.2308$, and $a_4 = \frac{5}{16} = 0.3125$. Hence, we guess that $\{a_n\}$ is increasing. To prove our guess, we write
\[ a_n < a_{n+1} \iff \frac{2n - 3}{3n + 4} < \frac{2(n + 1) - 3}{3(n + 1) + 4} \iff (2n - 3) \cdot (3n + 7) < (2n - 1) \cdot (3n + 4) \quad \text{(by cross-multiplication)} \]
\[ \iff 6n^2 - 9n + 14n - 21 < 6n^2 - 3n + 8n - 4 \iff -21 < -4. \]

Since the last statement is true, all the preceding statements are also true (as they are all equivalent). Hence, $a_n < a_{n+1}$, and we have proved that $\{a_n\}$ is increasing. (An alternative way to show that $\{a_n\}$ is increasing is to consider the corresponding function $f(x) = \frac{2x - 3}{3x + 4}$, and to prove that $f(x)$ is increasing for $x > 0$, i.e., that $f'(x) > 0$ holds:
\[ f'(x) = \frac{2 \cdot (3x + 4) - (2x - 3) \cdot 3}{(3x + 4)^2} = \frac{17}{(3x + 4)^2} > 0, \]
and since $a_n = f(n)$, the sequence $\{a_n\}$ is also increasing.)
(b) For \( \{a_n\} \) to be bounded, it has to be bounded both above and below. Since \( \{a_n\} \) is increasing, the smallest term in the sequence is the first term \( a_1 \). Hence, the sequence is bounded below by \( a_1 = \frac{-1}{1} \), i.e., \( a_n \geq \frac{-1}{1} \) for every \( n \geq 1 \). To show that the sequence is bounded above, we estimate

\[
a_n = \frac{2n - 3}{3n + 4} < \frac{2n - 3}{3n} < \frac{2n}{3n} = \frac{2}{3}
\]

for every \( n \geq 1 \).

Hence, the sequence is bounded above by \( \frac{2}{3} \). (Alternatively, one can argue that \( \lim_{n \to \infty} a_n = \frac{2}{3} \) and, hence, that the sequence is bounded above by its limit, since it is increasing and approaching that limit from below.)

**Problem 6 (15 points).**

Determine whether the statements below are true or false. If a statement is true, explain why; if it is false, explain why or give an example that disproves the statement.

(a) The function \( f(x) = \frac{\ln x}{x} \) is a solution of the differential equation \( x^2y' + xy = 1 \).

(b) The differential equation \( \frac{dy}{dx} = x - 2y \) is separable.

(c) Every monotonic sequence is convergent.

**Solution.**

(a) TRUE. To verify that \( y = f(x) = \frac{\ln x}{x} \) is a solution of the given equation, we have to plug \( y \) as well as \( y' \) into the equation and see whether it is satisfied. Compute

\[
y' = f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2};
\]

then,

\[
x^2y' + xy = x^2 \cdot \frac{1 - \ln x}{x^2} + x \cdot \frac{\ln x}{x} = 1 - \ln x + \ln x = 1,
\]

as required.

(b) FALSE. For a differential equation of the form \( \frac{dy}{dx} = F(x, y) \) to be separable, we have to be able to write its right-hand side as a product of two functions \( f(x) \) and \( g(y) \) which only depend on \( x \) and \( y \), respectively: \( \frac{dy}{dx} = f(x) \cdot g(y) \). In our case, however, the right-hand side can only be written as the difference of two such functions.

(c) FALSE. A sequence \( \{a_n\} \) can be monotonic (i.e., increasing or decreasing) but not converge to any limit if it increases or decreases without bound, i.e., if \( \lim_{n \to \infty} a_n = \pm \infty \). Counter-examples are e.g. \( a_n = n \) (increasing, but \( \lim_{n \to \infty} n = \infty \)) or \( a_n = -e^n \) (decreasing, but \( \lim_{n \to \infty} (-e^n) = -\infty \)).