Problem 1 (15 points).
Let the matrix \( A \) be given by
\[
\begin{bmatrix}
1 & -2 & -1 \\
-1 & 5 & 6 \\
5 & -4 & 5
\end{bmatrix}.
\]

(a) Find the inverse \( A^{-1} \) of \( A \), if it exists.

(b) Based on your answer in (a), determine whether the columns of \( A \) span \( \mathbb{R}^3 \). (Justify your answer!)

Solution.

(a) To check whether \( A \) is invertible, we row reduce the augmented matrix \([A \ I_3]: \)
\[
\begin{bmatrix}
1 & -2 & -1 & 1 & 0 & 0 \\
-1 & 5 & 6 & 0 & 1 & 0 \\
5 & -4 & 5 & 0 & 0 & 1
\end{bmatrix} \sim \ldots \sim \begin{bmatrix}
1 & -2 & -1 & 1 & 0 & 0 \\
0 & 3 & 5 & 1 & 1 & 0 \\
0 & 0 & 0 & -7 & -2 & 1
\end{bmatrix}.
\]
Since the last row in the echelon form of \( A \) contains only zeros, \( A \) is not row equivalent to \( I_3 \). Hence, \( A \) is not invertible, and \( A^{-1} \) does not exist.

(b) Since \( A \) is not invertible by (a), the Invertible Matrix Theorem says that the columns of \( A \) cannot span \( \mathbb{R}^3 \).

Problem 2 (15 points).
Let the vectors \( b_1, \ldots, b_4 \) be defined by
\[
b_1 = \begin{pmatrix} 3 \\ 5 \\ -2 \\ 4 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 7 \end{pmatrix}, \quad b_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad b_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}.
\]

(a) Determine if the set \( B = \{b_1, b_2, b_3, b_4\} \) is linearly independent by computing the determinant of the matrix \( B = [b_1 \ b_2 \ b_3 \ b_4] \).

(b) Using your answer in (a), determine if \( B \) is a basis for \( \mathbb{R}^4 \). (Justify your answer!)
Solution.

(a) The determinant of $B$ is most easily computed by first going down the fourth column,

$$\det B = \det \begin{bmatrix} 3 & 2 & -1 & 0 \\ 5 & -1 & 1 & 0 \\ -2 & -5 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{bmatrix} = (-1)^{4+4}(-3) \det \begin{bmatrix} 3 & 2 & -1 \\ 5 & -1 & 1 \\ -2 & -5 & 3 \end{bmatrix}.$$ 

Now, one possibility to compute the determinant of the $3 \times 3$-submatrix $B_{44}$ is

$$\det B_{44} = (3)(-1)(3) + (2)(1)(-2) + (-1)(5)(-5) - [(-2)(-1)(-1) + (-5)(1)(3) + (3)(5)(2)] = -1.$$ 

Hence, $\det B = (-3)(-1) = 3$.

(b) Since $\det B \neq 0$, it follows that the matrix $B$ is invertible. Hence, by the Invertible Matrix Theorem, the columns of $B$ are linearly independent, and the columns of $B$ span $\mathbb{R}^4$. Therefore, the set $B$ is a basis for $\mathbb{R}^4$.

Problem 3 (15 points).
Let the matrix $A$ be given by

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$ 

(a) Find a basis for the column space $\text{Col}A$ of $A$.

(b) Find a basis for the null space $\text{Nul}A$ of $A$.

(c) What are the dimensions of $\text{Col}A$ and $\text{Nul}A$? (Justify your answers!)

Solution.

(a) To find a basis for $\text{Col}A$, we have to reduce $A$ to echelon form:

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

The pivot columns in the echelon form are the first and third columns; therefore, a basis for $\text{Col}A$ is given by the first and third columns of $A$, $(-3, 1, 2)$ and $(-1, 2, 5)$.

(b) To obtain a basis for $\text{Nul}A$, we have to find the solution set of $Ax = 0$. Hence, we continue row reducing until the reduced echelon form of $A$ is found:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
The basic variables $x_1$ and $x_3$ can be expressed in terms of the free variables $x_2$, $x_4$, and $x_5$, with $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$. The general solution in parametric vector form is given by

$$
\mathbf{x} = \begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5
\end{pmatrix} = x_2 \begin{pmatrix}
 2 \\
 1 \\
 0 \\
 0 \\
 0
\end{pmatrix} + x_4 \begin{pmatrix}
 1 \\
 0 \\
 -2 \\
 1 \\
 0
\end{pmatrix} + x_5 \begin{pmatrix}
 -3 \\
 0 \\
 2 \\
 0 \\
 1
\end{pmatrix}.
$$

Hence, a basis for $\text{Nul} A$ is given by the three vectors $(2, 1, 0, 0, 0)$, $(1, 0, -2, 1, 0)$, and $(-3, 0, 2, 0, 1)$.

(c) Since the basis for $\text{Col} A$ found in (a) consists of two vectors, the dimension of $\text{Col} A$ is 2. Since the basis for $\text{Nul} A$ found in (b) consists of three vectors, the dimension of $\text{Nul} A$ is 3.

(Note: This agrees with the Rank Theorem, since $\dim(\text{Col} A) + \dim(\text{Nul} A) = 5$ equals the number of columns of $A$.)

**Problem 4 (15 points).**

Let $V$ be a vector space, and let $B = \{b_1, \ldots, b_n\}$ be a basis for $V$. Show that the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is an isomorphism from $V$ onto $\mathbb{R}^n$.

**Solution.**

To show that the coordinate mapping is an isomorphism, we have to show that it is linear, one-to-one, and onto. For vectors $\mathbf{x}$ and $\mathbf{y}$ in $V$, let $\mathbf{x} = c_1 b_1 + \ldots + c_n b_n$ and $\mathbf{y} = d_1 b_1 + \ldots + d_n b_n$. Then, $[\mathbf{x}]_B = (c_1, \ldots, c_n)$ and $[\mathbf{y}]_B = (d_1, \ldots, d_n)$. Moreover, $\mathbf{x} + \mathbf{y} = (c_1 + d_1) b_1 + \ldots + (c_n + d_n) b_n$, and

$$
[\mathbf{x} + \mathbf{y}]_B = (c_1 + d_1, \ldots, c_n + d_n) = (c_1, \ldots, c_n) + (d_1, \ldots, d_n)
$$

$$
= [\mathbf{x}]_B + [\mathbf{y}]_B.
$$

Also, $\mathbf{c} \mathbf{x} = (c c_1) b_1 + \ldots + (c c_n) b_n$, and

$$
[c \mathbf{x}]_B = (c c_1, \ldots, c c_n) = c (c_1, \ldots, c_n)
$$

$$
= c [\mathbf{x}]_B,
$$

and the coordinate mapping is therefore linear. To show that it is one-to-one, assume that $[\mathbf{x}]_B = (c_1, \ldots, c_n) = [\mathbf{y}]_B$ for two vectors $\mathbf{x}$ and $\mathbf{y}$ in $V$. Then,

$$
\mathbf{x} = c_1 b_1 + \ldots + c_n b_n \quad \text{and} \quad \mathbf{y} = c_1 b_1 + \ldots + c_n b_n,
$$

so $\mathbf{x} = \mathbf{y}$, which proves one-to-one-ness. To show that the coordinate mapping is onto, let $(c_1, \ldots, c_n)$ be a vector in $\mathbb{R}^n$. Then, $(c_1, \ldots, c_n)$ is the image of the vector $\mathbf{x} = c_1 b_1 + \ldots + c_n b_n$ in $V$, that is, $[\mathbf{x}]_B = (c_1, \ldots, c_n)$, which proves onto-ness.

(Note: To prove that $V$ and $\mathbb{R}^n$ are isomorphic for *general* vector spaces $V$, you cannot use the change-of-coordinates matrix $P_B = [b_1 \ldots b_n]$. This matrix is only defined if $V = \mathbb{R}^n$, that is, if the vectors $b_1, \ldots, b_n$ are vectors in $\mathbb{R}^n$ that can be written into a matrix!)
**Problem 5 (20 points).**

Let \( \mathbb{P} \) denote the vector space of all polynomials, and let \( \mathbb{P}_2 \) be the set of all polynomials of degree at most 2; that is, \( \mathbb{P}_2 = \{ p(t) : p(t) = a_0 + a_1 t + a_2 t^2, \ a_0, a_1, a_2 \text{ real} \} \).

(a) Show that \( \mathbb{P}_2 \) is a subspace of \( \mathbb{P} \).

(b) Using coordinate vectors, show that the set \( B \) given by

\[
B = \{ 1 + t^2, 2 - t + 3t^2, 1 + 2t - 4t^2 \}
\]

is a basis for \( \mathbb{P}_2 \).

(c) Find the coordinate vector \([p]_B\) of the polynomial \( p(t) = -4 - t^2 \) relative to \( B \).

(d) Find the polynomial \( q(t) \) whose coordinate vector relative to \( B \) is \([q]_B = (-3, 1, 2)\).

**Solution.**

(a) Since the zero polynomial \( p = 0 \) is obtained for \( a_0 = a_1 = a_2 = 0 \), \( \mathbb{P}_2 \) contains the zero vector of \( \mathbb{P} \). Given two polynomials \( p(t) = a_0 + a_1 t + a_2 t^2 \) and \( q(t) = b_0 + b_1 t + b_2 t^2 \) in \( \mathbb{P}_2 \), the sum

\[
(p + q)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2
\]

is in \( \mathbb{P}_2 \). Hence, \( \mathbb{P}_2 \) is closed under vector addition. Also, for any scalar \( c \),

\[
(c p)(t) = (ca_0) + (ca_1)t + (ca_2)t^2
\]

is in \( \mathbb{P}_2 \), and \( \mathbb{P}_2 \) is closed under scalar multiplication. So, in sum, \( \mathbb{P}_2 \) is a subspace of \( \mathbb{P} \).

(b) The coordinate vectors of the polynomials \( 1 + t^2, 2 - t + 3t^2, \) and \( 1 + 2t - 4t^2 \) are \((1, 0, 1), (2, -1, 3), \) and \((1, 2, -4), \) respectively. (The entries in the coordinate vectors contain the coefficients of \(1, t,\) and \(t^2,\) respectively.) Since the matrix formed from these vectors is row equivalent to the identity matrix \( I_3, \)

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 2 \\
1 & 3 & -4
\end{bmatrix}
\sim \ldots \sim
\begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & -2 \\
0 & 0 & -3
\end{bmatrix}
\sim \ldots \sim I_3,
\]

the coordinate vectors are linearly independent and span \( \mathbb{R}^3 \). By the isomorphism between \( \mathbb{P}_2 \) and \( \mathbb{R}^3, \) the corresponding polynomials \( 1 + t^2, 2 - t + 3t^2, \) and \( 1 + 2t - 4t^2 \) are linearly independent and span \( \mathbb{P}_2 \). Therefore, they form a basis for \( \mathbb{P}_2 \).

(c) To find \([p]_B,\) we have to determine how \( p(t) = -4 - t^2 = (-4)1 + (0)t + (-1)t^2 \) can be combined from the polynomials in \( B.\) This can be done by solving the linear system obtained from the corresponding coordinate vectors:

\[
\begin{bmatrix}
1 & 2 & 1 & -4 \\
0 & -1 & 2 & 0 \\
1 & 3 & -4 & -1
\end{bmatrix}
\sim \ldots \sim
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & -1
\end{bmatrix}.
\]
Hence, \([p]_B = (1, -2, 1)\).

(d) To find the polynomial \(q\) corresponding to \([q]_B = (-3, 1, 2)\), we just compute the matrix-vector product

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 2 \\
1 & 3 & -4
\end{bmatrix}
\begin{bmatrix}
-3 \\
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 \\
3 \\
-8
\end{bmatrix}.
\]

Therefore, \(q(t) = 1 + 3t - 8t^2\).

**Problem 6 (20 points).**

Determine whether the statements below are true or false. (*Justify* your answers: If a statement is true, explain why it is true; if it is false, explain why, or give a counter-example for which it is false.)

(a) If \(A\) and \(B\) are \(m \times n\)-matrices, then both \(AB^T\) and \(A^TB\) are defined.

(b) The determinant of an \(n \times n\)-matrix \(A\) is the product of the diagonal entries in \(A\).

(c) For \(A\) an \(m \times n\)-matrix, \(\text{Col}A\) is the set of all solutions of the linear system \(Ax = b\).

(d) For \(x\) in \(\mathbb{R}^n\), the coordinate vector \([x]_E\) of \(x\) relative to the standard basis \(E\) is \(x\) itself.

**Solution.**

(a) True. For \(A\) and \(B\) \(m \times n\), \(A^T\) and \(B^T\) are \(n \times m\), so both matrix products \(AB^T\) and \(A^TB\) are defined: The number of columns of \(A\), \(n\), equals the number of rows of \(B^T\); the number of columns of \(A^T\), \(m\), equals the number of rows of \(B\).

(b) False; this statement is only true if \(A\) is a triangular matrix. Take e.g.

\[
A = \begin{bmatrix}
1 & 2 \\
1 & 0
\end{bmatrix};
\]

then, \(\det A = (1)(0) - (1)(2) = -2\), whereas the product of the diagonal elements is \(0\).

(c) False. The column space \(\text{Col} A\) is the set of all linear combinations of the columns of \(A\), that is,

\[
\text{Col} A = \{ b \in \mathbb{R}^m : Ax = b \text{ for some } x \text{ in } \mathbb{R}^n \}.
\]

The solution set of \(Ax = b\) would be the set of all \(x\) in \(\mathbb{R}^n\) such that \(Ax = b\)!

(d) True. In general, the coordinate vector of \(x\) in \(\mathbb{R}^n\) relative to a basis \(B\) is related to \(x\) by \(P_B[x]_B = x\). For \(B = E\), the change-of-coordinates matrix \(P_E\) is the identity matrix \(I_n\). Hence, \(x = I_n[x]_E = [x]_E\).