1. Define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$, where

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}.$$ 

Find a basis $B$ for $\mathbb{R}^2$ with the property that $[T]_B$ is diagonal.

**Solution:** Diagonalize $A$ by finding the eigenvalues and eigenvectors of $A$: The characteristic polynomial is $\lambda^2 - 5\lambda = \lambda(\lambda - 5)$, so the eigenvalues of $A$ are $5$ and $0$. For $\lambda = 5$,

$$A - 5I_2 = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix},$$

and a basis for the eigenspace is thus $v_1 = (-2, 1)$. Similarly, for $\lambda = 0$, a basis is given by a solution to $Ax = 0$; one can take, e.g., $v_2 = (3, 1)$. From $v_1$ and $v_2$, we construct

$$P = [v_1 \ v_2] = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}.$$ 

By Theorem 8, the basis $B = \{v_1, v_2\}$ has the property that the $B$-matrix of $x \mapsto Ax$ is diagonal.

2. Define the matrix $C$ by

$$C = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}.$$ 

(a) Show that the eigenvalues of $C$ are $\sqrt{3} \pm 3i$, with corresponding eigenvectors $(1, \pm i)$.

(b) The transformation $x \mapsto Cx$ is the composition of a rotation and a scaling. Give the angle $\varphi$ of the rotation ($-\pi \leq \varphi \leq \pi$), and give the scaling factor $r$.

**Solution:** (a) Computing

$$Cx = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \sqrt{3} + 3i \\ -3 + \sqrt{3}i \end{bmatrix} = (\sqrt{3} + 3i) \begin{bmatrix} 1 \\ i \end{bmatrix},$$

one sees that $(1, i)$ is an eigenvector corresponding to the eigenvalue $\sqrt{3} + 3i$. Therefore, the complex conjugate vector $(1, -i)$ must also be an eigenvector corresponding to $\sqrt{3} - 3i$.

(b) The angle of rotation of the transformation is

$$\varphi = \arctan\left(\frac{-3}{\sqrt{3}}\right) = -\frac{\pi}{3};$$

the scaling factor is

$$r = |\lambda| = \sqrt{(\sqrt{3})^2 + (-3)^2} = 2\sqrt{3}.$$