1. Let

\[ A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}. \]

(a) Solve the initial value problem \( x'(t) = Ax(t) \) for \( t \geq 0 \), with \( x(0) = (1, 1) \).
(b) Classify the nature of the origin (attractor, repellor, or saddle point).
(c) Find the directions of greatest attraction, or repulsion, and sketch typical trajectories.

Solution: (a) Since \( \det(A - \lambda I_2) = \lambda^2 - 1 \), the eigenvalues of \( A \) are \( \lambda_1 = -1 \) and \( \lambda_2 = 1 \).

Bases for the corresponding two eigenspaces are for instance \( \mathbf{v}_1 = (-1, 1) \) and \( \mathbf{v}_2 = (-3, 1) \).

To find \( c_1 \) and \( c_2 \) such that \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = x(0) \), row reduce

\[
\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & x(0) \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.
\]

Hence, \( c_1 = 2 \) and \( c_2 = -1 \), and the solution of the initial-value problem is given by

\[ x(t) = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + (-1) \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t. \]

(b) Since one of the eigenvalues is negative and the other one is positive, the origin is a saddle point of \( x' = Ax \).

(c) The direction of greatest attraction is the line through \( \mathbf{v}_1 \) and the origin; the direction of greatest repulsion is the line through \( \mathbf{v}_2 \) and the origin.

2. Let the vectors \( \mathbf{u} \) and \( \mathbf{v} \) be defined by

\[ \mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -7 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -3 \\ 5 \\ 4 \\ 0 \end{pmatrix}. \]

(a) Determine if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal.
(b) Compute the distance \( \text{dist}(\mathbf{u}, \mathbf{v}) \) between \( \mathbf{u} \) and \( \mathbf{v} \).
(c) Find a unit vector in the direction of \( \mathbf{u} \).

Solution: (a) Since \( \mathbf{u} \cdot \mathbf{v} = 1(-3) + (-2)5 + 3(4) + (-7)0 = -1 \neq 0 \), \( \mathbf{u} \) and \( \mathbf{v} \) are not orthogonal.

(b) Since \( \mathbf{u} - \mathbf{v} = (4, -7, -1, -7) \),

\[ \text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-7)^2 + (-1)^2 + (-7)^2} = \sqrt{115} \approx 10.72. \]

(c) Since \( \|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 3^2 + (-7)^2} = \sqrt{63} = 3\sqrt{7} \), a unit vector in the direction of \( \mathbf{u} \) is given by

\[ \frac{1}{\|\mathbf{u}\|} \mathbf{u} = \frac{1}{3\sqrt{7}} \begin{pmatrix} 1 \\ -2 \\ 3 \\ -7 \end{pmatrix}. \]