1. Let \( H = \text{Span}\{v_1, v_2, v_3, v_4\} \), where

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, & \mathbf{v}_2 &= \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}, & \mathbf{v}_3 &= \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, & \mathbf{v}_4 &= \begin{pmatrix} -4 \\ -8 \\ 9 \end{pmatrix}.
\end{align*}
\]

(a) Explain why \( H \) is a subspace of \( \mathbb{R}^3 \). (Justify your answer!)
(b) Find a basis for \( H \).
(c) Using your answer in (b), explain why \( H \) is isomorphic to \( \mathbb{R}^2 \). (Justify your answer!)

Solution: (a) Since the span of a set of vectors in a vector space \( V \) always is a subspace of \( V \) (Theorem 1, Section 4.1) and since \( H = \text{Span}\{v_1, v_2, v_3, v_4\} \), \( H \) is a subspace of \( \mathbb{R}^3 \).

(b) Write the vectors \( v_1, \ldots, v_4 \) into a matrix \( A \) and then row reduce \( A \) to find its pivot columns:

\[
A = [v_1 \ldots v_4] = \begin{bmatrix}
1 & 6 & 2 & -4 \\
-3 & 2 & -2 & -8 \\
4 & -1 & 3 & 9
\end{bmatrix} \sim \ldots \sim \begin{bmatrix}
1 & 6 & 2 & -4 \\
0 & 5 & 1 & -5 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The first two columns of \( A \) are the pivot columns and hence form a basis of \( \text{Col} A = H \); in other words, \( B = \{v_1, v_2\} \) is a basis for \( H \).

(c) Since any vector \( x \) in \( H \) can be written as a linear combination of \( v_1 \) and \( v_2 \), that is, since

\[
x = c_1 v_1 + c_2 v_2
\]

for some scalars \( c_1, c_2 \), it follows that the coordinate vector \( [x]_B = (c_1, c_2) \) of \( x \) is in \( \mathbb{R}^2 \). Hence, \( H \) is isomorphic to \( \mathbb{R}^2 \) under the coordinate mapping, see Theorem 8 in Section 4.4.

2. Let

\[
\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}.
\]

(a) Show that the set \( B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \) is a basis for \( \mathbb{R}^3 \).
(b) Find the change-of-coordinate matrix \( P_B \) from \( B \) to the standard basis \( E \).
(c) Find the coordinate vector \( [x]_B \) of \( x = (-8, 2, 3) \) relative to \( B \).

Solution: (a) The change-of-coordinates matrix \( P_B = [\mathbf{b}_1 \, \mathbf{b}_2 \, \mathbf{b}_3] \) is row-equivalent to the identity matrix \( I_3 \). Hence, by the Invertible Matrix Theorem, \( P_B \) is invertible, and its
columns form a basis for $\mathbb{R}^3$.

(b) By (a), it follows that

$$P_B = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

for the change-of-coordinates matrix $P_B$, with $[x]_E = x = P_B [x]_B$.

(c) To solve the equation $[x]_B = P_B^{-1} x$ for $[x]_B$, row reduce the augmented matrix $[P_B \ x]$:

$$\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Hence, $[x]_B = (-5, 2, 1)$. 
