A geometric analysis of logarithmic switchback phenomena

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Abstract. One common characteristic of many classical singular perturbation problems is the occurrence of logarithmic (switchback) terms in the corresponding asymptotic expansions. We discuss two such problems well known to give rise to logarithmic switchback: first, Lagerstrom's equation, a model related to the asymptotic treatment of low Reynolds number flow from fluid mechanics, and second, the Evans function approach to the stability of degenerate shock waves in (scalar) reaction-diffusion equations. We show how asymptotic expansions for these two problems can be obtained by means of methods from dynamical systems theory as well as of the blow-up technique. We identify the structure of these expansions and demonstrate that the occurrence of the logarithmic switchback terms therein is in fact caused by a resonance phenomenon.

1. Introduction

A distinctive feature of asymptotic expansions in numerous singular perturbation problems is the occurrence of logarithmic terms. The structure of these expansions is both complicated and unexpected, as the governing equations typically give no hint of the presence of such terms. Traditionally, logarithmic terms have been accounted for under the notion of *switchback*, and resolved e.g. by the method of *matched asymptotic expansions*. There, the introduction of these terms is found not to be forced by the equations themselves, but by the matching process.

Our goal is to show how such expansions can be obtained using geometric methods, thereby establishing a connection with the classical approach. Our analysis is based on methods from the theory of dynamical systems, in particular on invariant manifolds and normal forms, and relies heavily on the so-called *blow-up technique*. Blow-up is essentially a sophisticated rescaling which allows one to analyze the dynamics near a singularity, cf. [1]. For instance, blow-up has been employed in [2] and [3] to give a geometric analysis of the singularly perturbed planar fold. Apart from deriving asymptotic expansions continued beyond the fold point, they also explained the structure of these expansions and gave an algorithm for the computation of their coefficients.

The present work is similar in spirit: here, we briefly discuss two classical singular perturbation problems well known to entail logarithmic switchback terms, and indicate how rigorous asymptotic expansions for these problems can be derived using geometric methods. Moreover, we identify the source of the switchback phenomenon in that we show that the occurrence of logarithmic terms in these expansions is caused by resonances in the respective blown-up equations. At this point, we conjecture that similar resonance phenomena are responsible for the occurrence of logarithmic terms in many other singular perturbation problems, at least after reinterpretation in a dynamical systems framework.

This article is organized as follows. In Section 2, we briefly discuss logarithmic switchback both from a classical point of view and from a geometric perspective. In Section 3, we introduce Lagerstrom's model equation and give a rough outline of the results published in full detail in [4, 5]. Section 4 contains a short survey of work in progress: we consider degenerate shock waves in (scalar) reaction-diffusion equations and indicate how Evans functions can still be meaningfully defined in that setting using blow-up [6].

2. Logarithmic switchback

A customary description of switchback would be the following [7]: let a perturbation problem involving a small parameter ε be given, and assume an expansion for the corresponding solution in terms of some sequence $\{\delta_j(\varepsilon)\}_{j\in\mathbb{N}}$, with $\delta_{j+1}(\varepsilon) = o(\delta_j(\varepsilon))$. After having computed the expansion to some order $\delta_k(\varepsilon)$, however, one finds it necessary to insert intermediate terms of the order $\delta^*(\varepsilon)$, where $\delta^* = o(\delta_k)$ and $\delta_{k+1} = o(\delta^*)$. To put it differently, at a certain step in the perturbation expansion, the neglected terms appear to be only $\mathcal{O}(\delta_{k+1})$, whereas their actual effect is $\mathcal{O}(\delta^*)$.

The above characterization is subjective, of course: one made a wrong guess about which expansion parameter to use; hence, one has to revert ("switch back") to the original expansion and insert additional terms therein a posteriori. If, moreover, these new terms involve logarithms of ε, the label logarithmic switchback is used.

More specifically, consider a system of ordinary differential equations of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \varepsilon) \qquad \text{with } \mathbf{x} \in \mathbb{R}^n \ (n \in \mathbb{N}) \text{ and } 0 < \varepsilon \ll 1 \tag{1}$$

now, and assume a naive expansion for the solution $\mathbf{x}(t, \varepsilon)$ to (1) of the form

$$\mathbf{x}(t, \varepsilon) = \mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + \dots \tag{2}$$

for t fixed as $\varepsilon \to 0$. Plugging (2) into (1) and comparing the coefficients of like powers of ε, one expects to find a recursive sequence of differential equations for $\{\mathbf{x}_j(t)\}_{j\in\mathbb{N}}$. In trying to solve these equations subject to a set of additional (boundary) constraints imposed on (1), however, one fails. In the simplest case, one might find that already in the second step of the expansion, the equation for $\mathbf{x}_1$ has no admissible solution. Hence, the assumption expressed by (2) is misleading: in the formal expansion, one expected the correction to $\mathbf{x}_0$ to be strictly $\mathcal{O}(\varepsilon)$, but the guess was wrong, as that correction is actually larger ($\mathcal{O}(\varepsilon \ln \varepsilon)$, for example).

Perturbation problems in which a small parameter ε (but not $\ln \varepsilon$) occurs in the formulation of the problem, whereas $\ln \varepsilon$ arises in the asymptotic expansion (as $\varepsilon^k (\ln \varepsilon)^\ell$, $(\ln \varepsilon)^{-k}$, or even $\varepsilon^k (\ln |\ln \varepsilon|)^{-\ell}$), have first been encountered in the resolution of paradoxes from fluid mechanics [8]. Logarithmic switchback, however, is tied neither to these paradoxes nor to fluid mechanics. It is not even tied to singular perturbation problems, as pointed out in [7]: even regular perturbation techniques may lead to logarithmic terms.

From a classical point of view, the question of switchback is connected to the behavior of solutions to (1) for t small. The theory of linear differential equations tells us whether to expect basic solutions which are logarithmic at $t = 0$ [9]. In the nonlinear case, the occurrence of logarithms depends on the nonlinearity of the equation: clearly, logarithmic terms arise through particular integrals of the form

$$\int^t \frac{ds}{s}.$$

In fact, it is argued in [7] that such terms are not unnatural, and that they can be obtained by continuity for certain classes of differential equations whose solutions, as well as the corresponding expansions around a singular point, vary continuously with a parameter.

In the following, we want to give an alternative characterization of logarithmic switchback based on the theory of dynamical systems or, more precisely, on the blow-up technique. In particular, it will turn out that logarithmic terms occur very naturally in this setting, and are due to resonances among certain eigenvalues associated with the blown-up equations.

Consider the extended system

$$\mathbf{y}' = \mathbf{g}(\mathbf{y}) \tag{3}$$

obtained from (1) by appending the (trivial) equation $\varepsilon' = 0$, where we have set $\mathbf{y} := (\mathbf{x}, \varepsilon)^T$ and $\mathbf{g}(\mathbf{y}) := (\mathbf{f}(\mathbf{y}), 0)^T$. Assume $\mathbf{y}^* = \mathbf{0}$ to be a degenerate equilibrium of (3), and let

$$y_1 = \bar{r}^{\alpha_1}\bar{y}_1,\ \dots,\ y_{n+1} = \bar{r}^{\alpha_{n+1}}\bar{y}_{n+1} \tag{4}$$

be a corresponding blow-up transformation by which the origin in (3) can be (partially) desingularized; here $(\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$. The blown-up vector field, which is induced by the vector field in (3), is best studied by introducing *charts* [1, 2]. These charts, which we call K_i, are defined by $\bar{y}_i = 1$ $(i = 1, \dots, n+1)$ in (4).

Remark 1 *For any object $\square$ in the original setting, we denote by $\overline{\square}$ the corresponding blown-up object; in chart K_i, the same object will appear as $\square_i$ in the following.*

The chart K_{n+1} corresponding to $\bar{y}_{n+1}$ $(= \bar{\varepsilon}) = 1$ is frequently referred to as the *rescaling chart*. It plays a crucial role in most blow-up analyses; however, the equations in K_{n+1} will contain $\varepsilon = r_{n+1}^{\alpha_{n+1}}$ only as a parameter, whereas ε has to appear as a dynamic variable for switchback terms to occur, see below.

Hence, fix a value of $i \in \{1, \dots, n\}$ now, and let K_i denote the corresponding (*phase-directional*) chart in (4), with $\bar{y}_i = 1$. Without loss of generality, we assume that the origin is still an equilibrium in K_i. Then, the resulting equations in K_i, which are obtained by plugging (4) into (3) and expanding, can be written as

$$\mathbf{y}_i' = D\mathbf{g}_i(\mathbf{0})\,\mathbf{y}_i + \mathcal{O}(\mathbf{y}_i^2) \tag{5}$$

after desingularization. The desingularization, which corresponds to introducing a rescaled time t_i via $dt_i = r_i^{\alpha} dt$ $(\alpha \in \mathbb{N})$, is necessary to obtain non-trivial dynamics on $\{r_i = 0\}$.

For the sake of exposition, assume $D\mathbf{g}_i(\mathbf{0}) = \mathrm{diag}(\lambda_1, \dots, \lambda_{n+1})$ in (5), where $\{\lambda_j\}_{j=1}^{n+1}$ are distinct real numbers. Our goal is to show that if the eigenvalues of $D\mathbf{g}_i(\mathbf{0})$ are in *resonance*, i.e. if for some $k \in \{1, \dots, n+1\}$,

$$\lambda_k = \sum_{j=1}^{n+1} \alpha_j \lambda_j \qquad \text{with } \{\alpha_j\}_{j=1}^{n+1} \in \mathbb{N}^{n+1} \text{ and } \sum_{j=1}^{n+1} \alpha_j \geq 2, \tag{6}$$

this resonance will give rise to logarithmic terms in the corresponding expansions for $\mathbf{y}_i$. To keep the notation compact, we will suppress the subscript i throughout the remainder of this section.

Note that by assumption, $y_j \sim \beta_j e^{\lambda_j t}$ $(j = 1, \dots, n+1)$ in K, where $\{\beta_j\}_{j=1}^{n+1} \in \mathbb{R}^{n+1}$. Hence, (5) and (6) imply that to leading order,

$$y_k' = \lambda_k y_k + \beta e^{\sum_{j=1}^{n+1} \alpha_j \lambda_j t} = \lambda_k y_k + \beta e^{\lambda_k t} \qquad \text{with } \beta = \prod_{j=1}^{n+1} \beta_j^{\alpha_j}, \tag{7}$$

where by standard normal form theory we may assume that all non-resonant terms have been removed by a sequence of near-identity transformations [10]. A particular solution to (7) is given by

$$y_k = \beta t e^{\lambda_k t}. \tag{8}$$

Finally, to express y_k in in terms of ε, we recall $y_{n+1} = \varepsilon$ and $\varepsilon \sim \beta_{n+1} e^{\lambda_{n+1} t}$. Therefore, plugging $t \sim \frac{1}{\lambda_{n+1}} \ln \frac{\varepsilon}{\beta_{n+1}}$ into (8), we find that $te^{\lambda_k t}$ gives rise to $\varepsilon \ln \varepsilon$ in y_k after elimination of t. Similarly, higher powers of $te^{\lambda_k t}$ in (7) would give rise to higher-order terms in $\varepsilon \ln \varepsilon$.

Thus, from a dynamical systems point of view, logarithmic switchback can in principle be expected in problems in which the source of a singularity is tied to degenerate equilibria which are amenable to a blow-up analysis and which display resonance in one of the phase-directional charts after blow-up. In particular, the above argument shows that the powers of ε and $\ln \varepsilon$ occurring in the resulting asymptotic expansions (after blow-down) are determined by the values of the resonant eigenvalues and their respective ratios, see [3, 5, 6] for examples.

3. Lagerstrom's model equation

A classical singular perturbation problem from fluid mechanics occurs in the asymptotic treatment of viscous flow past a solid at low Reynolds number, see e.g. [8]. Though attempts date back to [11], the conceptual structure of the problem was finally resolved in [12] and [13]. To illustrate the ideas and techniques used by Kaplun, Lagerstrom [9] proposed an analytically rather simple model problem which, in its simplest formulation, is given by

$$u'' + \frac{n-1}{\xi} u' + \varepsilon u u' = 0 \tag{9}$$

$$u(\xi = 1) = 0, \qquad u(\xi = \infty) = 1. \tag{10}$$

Here $n \in \mathbb{N}$, $0 \leq \varepsilon \ll 1$ (where ε is the analog of the Reynolds number), $1 \leq \xi \leq \infty$, and the prime denotes differentiation with respect to ξ. Lagerstrom's analysis of (9),(10) is based on the method of matched asymptotic expansions. There, it turns out that similar difficulties as in the original problem arise (*Stokes' paradox*, *Whitehead's paradox*).

In the following, we will only consider the physically relevant case $n = 3$ in (9). It is well known that for $n = 3$, an expansion for $v_\varepsilon := u'|_{\xi=1}$ is given by [9]

$$v_\varepsilon = 1 - \varepsilon \ln \varepsilon - (\gamma + 1)\varepsilon + \mathcal{O}(\varepsilon^2); \tag{11}$$

here $\gamma \approx 0.5772$ is Euler's constant. Incidentally, note that v_ε corresponds to the drag, a quantity of interest in the original fluid dynamical problem. For more background information and further references on Lagerstrom's model equation, we refer to [4].

3.1. Our approach

Keeping in mind that $n = 3$ now, we rewrite (9),(10) as a dynamical system in extended phase space by replacing $\xi \in [1, \infty)$ by $\eta := \xi^{-1} \in (0, 1]$, setting $u' = v$, and appending the (trivial) equation $\varepsilon' = 0$:

$$\begin{aligned} u' &= v \\ v' &= -2\eta v - \varepsilon u v \\ \eta' &= -\eta^2 \\ \varepsilon' &= 0 \end{aligned} \tag{12}$$

$$u(\xi = 1) = 0, \quad \eta(\xi = 1) = 1, \quad u(\xi = \infty) = 1. \tag{13}$$

For $\varepsilon > 0$ fixed, let $\mathcal{V}_\varepsilon$ be defined by $\mathcal{V}_\varepsilon := \{(0, v, 1) \mid v \in [\underline{v}, \overline{v}]\}$, where $0 \leq \underline{v} < \overline{v} < \infty$, and let the point Q be given by $Q = (1, 0, 0)$. Note that $\mathcal{V}_\varepsilon$ and Q correspond to the inner and outer boundary conditions in (13), respectively.

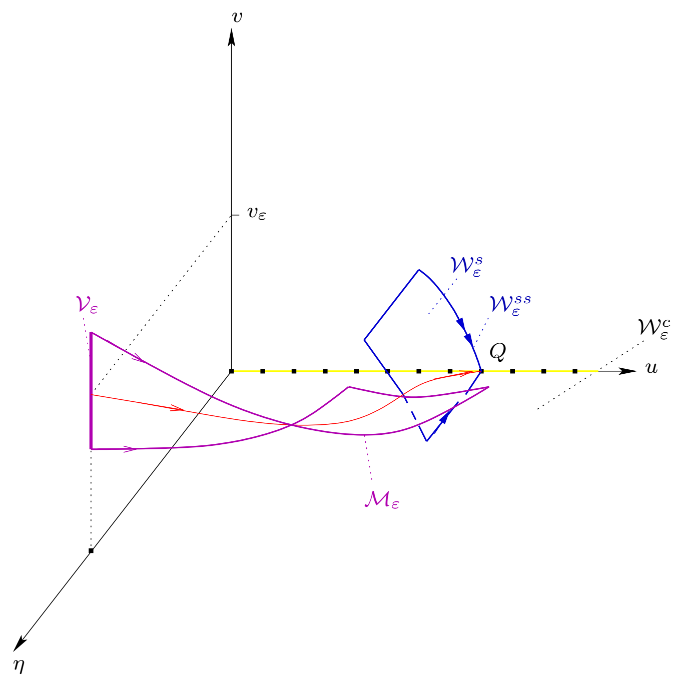

Figure 1. Geometry of system (12) for $\varepsilon > 0$ fixed.

The equilibria of (12) are located on the line $\ell := \{(u, 0, 0) \,|\, u \in \mathbb{R}^+\}$, with $\varepsilon \in [0, \varepsilon_0]$ and $\varepsilon_0 > 0$ small; obviously, $Q \in \ell$. Linearization at ℓ shows 0 to be a triple eigenvalue even for $\varepsilon \neq 0$. Still, the existence of one strongly stable direction $\mathcal{W}^{ss}_\varepsilon$ can be exploited to define a two-dimensional stable manifold $\mathcal{W}^s_\varepsilon$ for Q. The situation is illustrated in Figure 1.

Remark 2 *Although one cannot deduce the existence of $\mathcal{W}^s_0$ from standard invariant manifold theory, it is shown in [4] that $\mathcal{W}^s_\varepsilon$ can still be defined down to $\varepsilon = 0$ by means of blow-up.*

The (polar) blow-up transformation introduced in [4] to analyze the dynamics of (12) near ℓ is given by

$$u = \bar{u}, \quad v = \bar{r}\bar{v}, \quad \eta = \bar{r}\bar{\eta}, \quad \varepsilon = \bar{r}\bar{\varepsilon}.$$

Apart from the rescaling chart K_2 corresponding to $\bar{\varepsilon} = 1$, one requires an additional chart K_1 defined by $\bar{\eta} = 1$ here.

After transformation to K_1 (and desingularization by rescaling ξ with a factor of r_1), the equations in (12) become

$$\begin{aligned} u_1' &= v_1 \\ v_1' &= -v_1 - \varepsilon_1 u_1 v_1 \\ r_1' &= -r_1 \\ \varepsilon_1' &= \varepsilon_1. \end{aligned} \tag{14}$$

The equilibria of (14) lie in $\ell_1 := \{(u_1, 0, 0, 0) \,|\, u_1 \in \mathbb{R}^+\}$; a simple computation shows that the corresponding eigenvalues are given by -1 (double), 0, and 1. Since $0 = 1 + (-1)$, there is a resonance of order 2 among these eigenvalues.

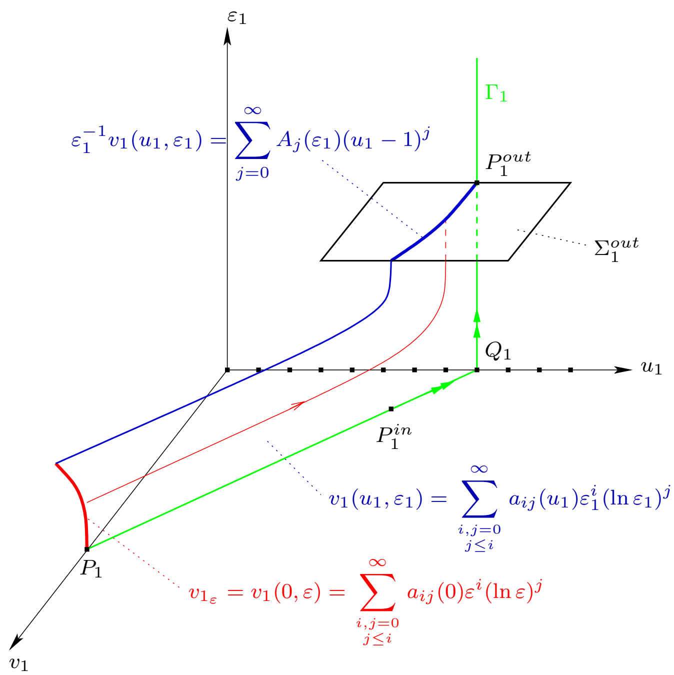

Figure 2. Expansions for v_ε in chart K_1 $(n = 3)$.

3.2. Rigorous asymptotic expansions

It is proved in [5] that $v_1 = v_1(u_1, \varepsilon_1)$ can be expanded as

$$v_1(u_1, \varepsilon_1) = \sum_{\substack{i,j=0 \\ j \leq i}}^{\infty} a_{ij}(u_1)\varepsilon_1^i (\ln \varepsilon_1)^j, \tag{15}$$

with unique and smooth coefficient functions $a_{ij}(u_1)$, see Figure 2. In fact, the complicated structure of (15) (and hence of (11)) is due to the passage past ℓ_1 in K_1. Heuristically, this can be understood as follows: by introducing the new variable $\tilde{v}_1 = e^{\xi_1} v_1$ in (14) and integrating, one obtains

$$\begin{aligned} u_1(\xi_1) &= u_{1_0} + \int_0^{\xi_1} e^{-\eta}\tilde{v}_1(\eta)\, d\eta \\ \tilde{v}_1(\xi_1) &= v_{1_0} - \varepsilon_{1_0} \int_0^{\xi_1} e^{\eta} u_1(\eta)\tilde{v}_1(\eta)\, d\eta, \end{aligned} \tag{16}$$

where we have used $\varepsilon_1 = \varepsilon_{1_0} e^{\xi_1}$ and u_{1_0}, v_{1_0}, and ε_{1_0} are constants. As it turns out, (16) defines a contraction operator for u_1 and $\tilde{v}_1$ in $\mathcal{L}^\infty[0, \ln \frac{\varepsilon_1}{\varepsilon_{1_0}}]$. Hence, a Picard iteration scheme can be applied, with the starting point given by $(u_1^{(0)}, \tilde{v}_1^{(0)}) = (u_{1_0}, v_{1_0})$:

$$u_1^{(1)} = u_{1_0} + v_{1_0}(1 - e^{-\xi_1}) \tag{17}$$

$$\tilde{v}_1^{(1)} = v_{1_0} + \varepsilon_{1_0} u_{1_0} v_{1_0}(1 - e^{-\xi_1}) \tag{18}$$

$$u_1^{(2)} = u_1^{(1)} + \varepsilon_{1_0} u_{1_0} v_{1_0}(1 - \xi_1 - e^{-\xi_1}) \tag{19}$$

$$\vdots$$

Now, since $\xi_1 = \ln \frac{\varepsilon_1}{\varepsilon_{1_0}}$, we find that there is a logarithmic switchback term in ε_1 when (19) is rewritten as a function of ε_1. Similarly, the products of powers of ξ_1 and e^{ξ_1} which occur for higher iterates in (16) will give rise to products of powers of $\ln \varepsilon_1$ and ε_1.

Remark 3 *Let $\Pi: \Sigma_1^{in} \to \Sigma_1^{out}$ denote the (local)* transition map *for (14), where Σ_1^{in} and Σ_1^{out} are appropriately defined sections in K_1. Then, the above computation gives the leading order behavior of Π.*

Remark 4 *The case $n = 2$ is computationally more involved, see [5] for details.*

4. Evans functions for characteristic shock waves

In our second example, we study the stability of viscous shock waves in the family of reaction-diffusion equations defined by

$$u_t + f(u)_x = u_{xx}. \tag{20}$$

Here $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $u(t, x) \in \mathbb{R}$, and $f: U \to \mathbb{R}$ is assumed to be smooth in $U \subset \mathbb{R}$, with U convex and open. Moreover, we require that $f(0) = 0$, $f(1) = 0$, and $f(u) > 0$ for $u \in (0, 1)$.

Remark 5 *Although we will restrict ourselves to the scalar case in the following, the results of this section can be generalized to systems with $u(t, x) \in \mathbb{R}^d$ for $d \in \mathbb{N}$, see the forthcoming article [6].*

Let $\phi(x)$ be a stationary shock wave for (20), with

$$\lim_{x\to-\infty} \phi(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} \phi(x) = 1.$$

Without loss of generality, we assume $f'(1) < 0$ and $f'(0) = 0$ in (20), which corresponds to a *characteristic shock* [14] with only algebraic decay of ϕ at $x = -\infty$. Therefore, we may write

$$f(u) = \frac{1}{2}u^2 + \frac{\alpha}{3}u^3 + \mathcal{O}(u^4)$$

with some $\alpha < 0$, possibly up to a rescaling of u. The requirement that $\alpha \neq 0$ is known as the *genuine nonlinearity assumption.*

Spectral stability of ϕ is given if the linearization of (20) around ϕ, which is

$$\mathcal{L}p := p_{xx} - (df(\phi)\,p)_x = \lambda p \qquad \text{with } \lambda \in \mathbb{C}, \tag{21}$$

has no eigenvalue $\lambda \neq 0$ in $\{\Re\lambda \geq 0\}$; here, we have set $u(t, x) = \phi(x) + p(t, x)$. To investigate the stability properties of ϕ, one commonly introduces a function $\mathcal{E}(\lambda)$, called the *Evans function*, such that zeros of $\mathcal{E}$ correspond to eigenvalues of $\mathcal{L}$. In contrast to the well-understood non-characteristic case, however, $\mathcal{E}$ is not analytic, but has a branch point at $\lambda = 0$. Details and further references can e.g. be found in [15, 16].

4.1. Our approach

As we are interested in shock wave solutions for (20), we introduce $\xi = x - ct$ in (20) and integrate once to obtain $u' = f(u)$. Moreover, we rewrite (21) as

$$\begin{aligned} p' &= df(u)\,p + \lambda q \\ q' &= p. \end{aligned}$$

Here, the prime denotes differentiation with respect to ξ, and we have shifted to a moving frame in which c equals zero. Introducing the complex projective coordinate $z := \frac{p}{q} \in \mathcal{P}(\mathbb{C})$, setting $\lambda = \gamma^2 \in \mathbb{C}$, and appending the (trivial) equation $\gamma' = 0$, we obtain

$$\begin{aligned} u' &= f(u) \\ z' &= df(u)\, z - z^2 + \gamma^2 \\ \gamma' &= 0. \end{aligned} \tag{22}$$

There are two lines $\ell^{\pm}$ of non-hyperbolic equilibria for (22), where $\ell^{\pm} = \{(0, \pm\gamma, \gamma) \,|\, \gamma \in \mathbb{C}\}$. The point $(0, 0, 0)$ corresponding to $\gamma\ (= \lambda) = 0$ is particularly degenerate, with 0 a triple eigenvalue.

To desingularize the origin in (22), we introduce the blow-up transformation

$$u = \bar{r}\bar{u}, \quad z = \bar{r}\bar{z}, \quad \gamma = \bar{r}\bar{\gamma},$$

where we assume $\bar{z}, \bar{\gamma} := \bar{\rho}e^{i\varphi} \in \mathbb{C}$. Two charts ($K_1$ and K_2) are required here, defined by $\bar{u} = 1$ and $\bar{\rho}\ (= |\bar{\gamma}|) = 1$.

The desingularized equations in chart K_1 are given by

$$\begin{aligned} r_1' &= (\tfrac{1}{2} + \tfrac{\alpha}{3} r_1 + \mathcal{O}(r_1^2)) r_1 \\ z_1' &= -(\tfrac{1}{2} + \tfrac{\alpha}{3} r_1 + \mathcal{O}(r_1^2)) z_1 + (1 + \alpha r_1 + \mathcal{O}(r_1^2)) z_1 - z_1^2 + \gamma_1^2 \\ \gamma_1' &= -(\tfrac{1}{2} + \tfrac{\alpha}{3} r_1 + \mathcal{O}(r_1^2)) \gamma_1. \end{aligned} \tag{23}$$

While $(0, 0, 0)$ is still an equilibrium for (23), there is now an additional equilibrium at $P_1 := (0, \frac{1}{2}, 0)$. The corresponding eigenvalues are $-\frac{1}{2}$ and $\frac{1}{2}$ (double) for the origin and $-\frac{1}{2}$ (double) and $\frac{1}{2}$ for P_1, respectively. Therefore, the analysis in K_1 is complicated by the fact that both these equilibria are resonant.

4.2. The Dulac map

To analyze the dynamics of (23) about P_1 in more detail, we define $\tilde{z}_1 = z_1 - \frac{1}{2}$ and then divide out the common factor $\frac{1}{2} + \frac{\alpha}{3} r_1 + \mathcal{O}(r_1^2)$ from the resulting equations:

$$\begin{aligned} r_1' &= r_1 \\ \tilde{z}_1' &= -\tilde{z}_1 + \frac{\alpha(\tilde{z}_1 + \frac{1}{3}) r_1 - \tilde{z}_1^2 + \gamma_1^2 + \mathcal{O}(r_1^2)}{\frac{1}{2} + \frac{\alpha}{3} r_1 + \mathcal{O}(r_1^2)} \\ \gamma_1' &= -\gamma_1. \end{aligned} \tag{24}$$

We introduce two sections Σ_1^{in} and Σ_1^{out} in K_1, as shown in Figure 3. An expansion for the (local) transition map $\Pi : \Sigma_1^{in} \to \Sigma_1^{out}$, $(r_1^{out}, \tilde{z}_1^{out}, \gamma_1^{out}) = \Pi(r_1^{in}, \tilde{z}_1^{in}, \gamma_1^{in})$, for (24) can be obtained from normal form theory [10], which gives

$$\tilde{z}^{out} = r_1^{out} \tilde{z}_1^{out} = \frac{\gamma}{\gamma_1^*} \tilde{z}_1^{in} + \frac{8\alpha}{3} \gamma^2 \log \gamma + \mathcal{O}(|\gamma|^2, |\gamma|^3 |\log \gamma|^2, |\gamma|^3 |\log \gamma|) \tag{25}$$

for some $0 \neq \gamma_1^* \in \mathbb{C}$ with $|\gamma_1^*|$ small. Again, one finds that the logarithmic terms in (25) are caused by resonance, as the eigenvalues of (24) are given by -1 and 1, cf. [6] for details. Note that the case $\alpha = 0$ has been excluded by definition, implying that the $\gamma^2 \log \gamma$-term in (25) does not vanish.

Remark 6 *By appropriately defining an Evans function $\mathcal{E}$ for* (22)*, see [17], one obtains from* (25) *that*

$$\frac{d}{d\gamma} \mathcal{E}(\gamma, \log \gamma) \Big|_{\gamma = 0} = \frac{d\tilde{z}^{out}}{d\gamma}(0) \neq 0;$$

therefore, $\gamma = 0$ is a simple root of $\mathcal{E}$, and $\mathcal{E}$ cannot be analytic for γ (and hence λ) close to zero.

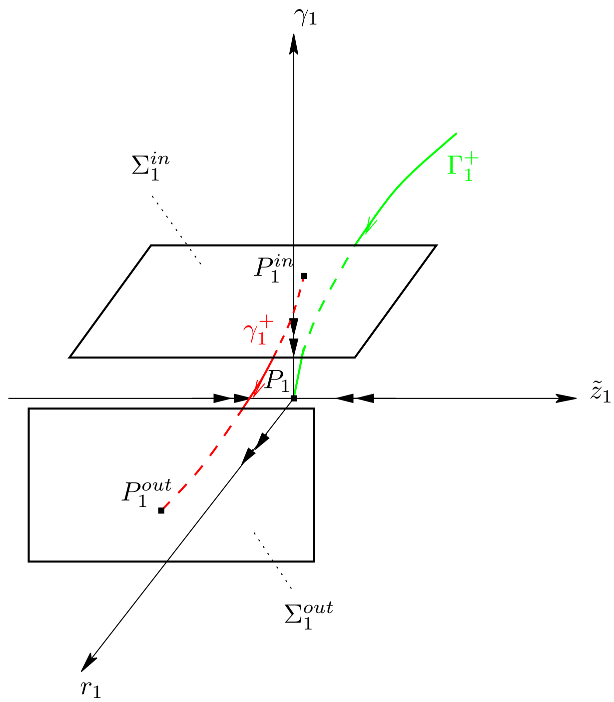

Figure 3. The Dulac map in chart K_1.

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