THE CRITICAL WAVE SPEED FOR THE FISHER-KOLMOGOROV-PETROWSKII-PISCOUNOV EQUATION WITH CUT-OFF

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ABSTRACT. The Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation with cut-off was introduced in [E. Brunet and B. Derrida, Shift in the velocity of a front due to a cutoff, *Phys. Rev. E* **56**(3), 2597–2604 (1997)] to model N-particle systems in which concentrations less than $\varepsilon = \frac{1}{N}$ are not attainable. It was conjectured that the cut-off function, which sets the reaction terms to zero if the concentration is below the small threshold ε, introduces a substantial shift in the propagation speed of the corresponding traveling waves. In this article, we prove the conjecture of Brunet and Derrida, showing that the speed of propagation is given by $c_{\rm crit}(\varepsilon) = 2 - \frac{\pi^2}{(\ln\varepsilon)^2} + \mathcal{O}((\ln\varepsilon)^{-3})$, as $\varepsilon \to 0$, for a large class of cut-off functions. Moreover, we extend this result to a more general family of scalar reaction-diffusion equations with cut-off. The main mathematical techniques used in our proof are geometric singular perturbation theory and the blow-up method, which lead naturally to the identification of the reasons for the logarithmic dependence of $c_{\rm crit}$ on ε as well as for the universality of the corresponding leading-order coefficient (π^2).

1. Introduction

The scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^2), \tag{1}$$

which is commonly known as the Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation [20, 24] and which also goes under the name Allen-Cahn equation [1], arises in numerous problems in biology, optics, combustion, and various other disciplines, see e.g. [8, 2, 3]. In particular, the dynamics of traveling wave solutions are frequently of interest in applications. In the context of (1), traveling waves are solutions that are stationary in a frame moving at the constant speed $c \geq 0$ and that connect the rest states $u = 0$ and $u = 1$. It is well-known that these waves exist for all $c \geq 2$; the speed $c_{\rm FKPP} = 2$, which is selected via classical stability considerations, is often referred to as the "critical" wave speed.

Here, we are interested especially in classes of problems in which the FKPP equation is obtained either in the large-scale limit ($N \to \infty$) of many-particle systems or in the mean-field limit of physical problems that are discrete at a microscopic level [6, 22, 7, 11, 12, 23].

The propagation speed of traveling waves in the continuum limit—$c_{\rm FKPP} = 2$ for the FKPP equation (1)—is often used to approximate the speed of propagation in such systems with large N. However, systematic numerical simulations have revealed that the observed speeds of propagation in these and other related systems are typically substantially smaller than expected [22, 7, 11, 23, 27] and that the characteristic propagation speed converges only very slowly to $c_{\rm FKPP}$ as $N \to \infty$. Even for large values of N, such as $N = 10^4$, the discrepancies are substantial.

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These numerical observations motivated Brunet and Derrida [9] to introduce the following modified or "cut-off" FKPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^2)\varphi(u), \tag{2}$$

where they assumed that the cut-off function φ satisfies

$$\varphi(u) \ll 1 \quad \text{if } u \ll \varepsilon \quad \text{and} \quad \varphi(u) \equiv 1 \quad \text{if } u > \varepsilon \tag{3}$$

for $\varepsilon \geq 0$ and small. The motivation in [9] was that, for any fixed value of N, no particles are available to react at any point in the domain at which the concentration is less than $\varepsilon = \frac{1}{N}$, and, hence, that the reaction function must be "cut-off" there.

One question addressed in [9] is precisely the effect of the cut-off function φ on the velocity c of the corresponding traveling fronts. In particular, for $\varphi(u) = \Theta(u-\varepsilon)$, where Θ denotes the Heaviside step function, it is shown in [9] that the front velocity c in (2) differs from the continuum wave speed c_{FKPP} by a term that is logarithmic in ε,

$$c \sim 2 - \frac{\pi^2}{(\ln \varepsilon)^2} \qquad \text{as } \varepsilon \to 0. \tag{4}$$

This leading-order approximation implies that, for $\varepsilon > 0$, a cut-off is introduced in the "tail" of the traveling front which decreases the front speed considerably and that the convergence as $\varepsilon \to 0$ to the unperturbed critical velocity c_{FKPP} is indeed slow. Moreover, it agrees well with the numerical data in [9].

Additionally, it is postulated in [9] that the choice of a specific cut-off function φ does not fundamentally influence the asymptotic behavior of (2) as long as (3) holds. It is conjectured that the leading-order correction to c will be *independent* of φ within this class of cut-off functions.

The main results of this article are a rigorous, geometric derivation of the expansion in (4) for $\varphi = \Theta$ (the Heaviside cut-off), as well as a proof of Brunet and Derrida's conjecture for a very general family of cut-off functions that do not even have to satisfy the condition that $\varphi(u) \ll 1$ when $u \ll \varepsilon$ put forward in [9], cf. (3). The proofs presented here are constructive and give precise information on the structure of the corresponding traveling waves. Moreover, in proving these results, we also provide clear geometric reasons for why the leading-order correction to c_{FKPP} is inversely proportional to the square of the logarithm of ε, and for why the corresponding coefficient (π^2) is universal within this class of cut-off functions. (See also Remark 7 below for a heuristic derivation of these results.)

In the present article, we will be concerned with the more general modified FKPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u^2)\varphi(u, \varepsilon, \tfrac{u}{\varepsilon}), \tag{5}$$

where we assume that the cut-off function φ satisfies the following conditions:

Assumption $\mathcal{A}$. For $k \geq 1$, there exists a $\mathcal{C}^k$-smooth function $\psi(u, \varepsilon, \cdot)$ which depends on u, ε, and a third (real) variable such that

$$\varphi(u, \varepsilon, \tfrac{u}{\varepsilon}) = \psi(u, \varepsilon, \tfrac{u}{\varepsilon}) \quad \text{if } u < \varepsilon \quad \text{and} \quad \varphi(u, \varepsilon, \tfrac{u}{\varepsilon}) \equiv 1 \quad \text{if } u > \varepsilon;$$

moreover, φ is bounded at $u = \varepsilon$, for ε sufficiently small. The function ψ is defined in some neighborhood of $\{0\} \times \{0\} \times [0,1]$ and satisfies $\psi(0,0,\cdot) \in [0,1]$ on $[0,1]$ as well as $\psi(0,0,0) \in [0,1)$.

For convenience, we write $\Psi(\cdot) := \psi(0,0,\cdot)$, and we observe that Assumption $\mathcal{A}$ generalizes the assumptions of [9] insofar as we do not require $\Psi(0) \ll 1$ here. (In other words, we may allow φ to be even only slightly smaller than one at the origin.)

***Remark* 1.** The particular choice $\varphi(u, \varepsilon, \frac{u}{\varepsilon}) = \Theta(u - \varepsilon)$ (the Heaviside cut-off) satisfies Assumption $\mathcal{A}$ with $\psi(u, \varepsilon, \cdot) \equiv 0$, as do the other choices for φ mentioned in [9]; for $\psi(u, \varepsilon, \cdot) = \cdot$, e.g., one obtains the linear cut-off, with $\varphi(u, \varepsilon, \frac{u}{\varepsilon}) = \frac{u}{\varepsilon}$ for $u \leq \varepsilon$. Note that φ will be continuous (and even Lipschitz continuous) at $u = \varepsilon$ if $\varphi(\varepsilon, \varepsilon, 1) = 1 = \psi(\varepsilon, \varepsilon, 1)$. This continuity is not a requirement in Assumption $\mathcal{A}$, though; the function $\varphi = \Theta$, for instance, is discontinuous at $u = \varepsilon$.

Correspondingly, (5) will not necessarily have classical (differentiable) solutions across $\{u = \varepsilon\}$. However, as is customary in this situation, we will only consider solutions of (5) that are continuous at $u = \varepsilon$, and that are therefore classical solutions for $u \geq \varepsilon$, respectively for $u \leq \varepsilon$, with $\varphi \equiv \psi$ in $\{u \leq \varepsilon\}$ and $\varphi \equiv 1$ in $\{u \geq \varepsilon\}$, respectively. Finally, we note that the exact value of $\varphi(\varepsilon, \varepsilon, 1)$ is insignificant; to be precise, φ can be interpreted as an equivalence class of bounded functions, with two such functions being equivalent if they only differ on $\{u = \varepsilon\}$.

Traditionally, the "critical" wave speed c_{crit} in scalar reaction-diffusion equations of FKPP-type is defined as the particular value of c that separates traveling wave solutions of different decay rates (exponential versus algebraic or even exponential versus algebro-exponential) at the zero rest state. In our case, however, a distinction has to be made depending on the behavior of Ψ near the origin, which naturally leads to a slightly generalized notion of "criticality" that encompasses the classical definition. In particular, we have the following three cases:

(i) If $\Psi(0) = 0$ and if, moreover, 0 is an isolated zero of Ψ, then there is an infinite semi-axis of wave speeds, with $0 < c_{\text{crit}} \leq c$, for which traveling wave solutions to (5) exist. The traveling wave corresponding to $c = c_{\text{crit}}$ decays exponentially at zero, whereas the decay is merely algebraic for $c > c_{\text{crit}}$.

(ii) If $\Psi(0) = 0$, but if, additionally, 0 is an accumulation point of positive zeros of Ψ, e.g. when Ψ vanishes in a neighborhood of 0, then there is a traveling wave solution to (5) for precisely one value of c.

(iii) If $\Psi(0) > 0$, then traveling wave solutions to (5) again exist for any $c \geq c_{\text{crit}}$, as in (i). However, the definition of a "critical" wave speed has to be generalized insofar as all traveling waves now decay exponentially at zero. The wave corresponding to $c = c_{\text{crit}}$ is distinguished due to the fact that its decay rate is the strongest, whereas for $c > c_{\text{crit}}$ solutions will decay at a weaker exponential rate.

Case (i) corresponds to the traditional definition of criticality, while (ii) is significant insofar as it includes e.g. the Heaviside cut-off $\varphi = \Theta$ analyzed in [9]. Moreover, we note that the notion of criticality is vacuous in that case, since traveling wave solutions exist only for one value of c; however, to ensure consistency of notation, we will still denote that value by c_{crit}. For the sake of exposition and because of the specific importance of that case, we will, in our analysis, often first consider case (ii), with the additional condition that $\Psi \equiv 0$ on $[0, 1]$. Then, we will carefully show that our arguments are also valid in all other cases under consideration. Finally, we note that in cases (i) and (iii), the critical wave speed can equivalently be characterized as the minimum value of c in (5) for which monotone traveling wave solutions with $u \in [0, 1]$ exist.

Given $\varepsilon \geq 0$ small, let $c_{\text{crit}}(\varepsilon)$ denote the corresponding critical wave speed for the cut-off FKPP equation (5) in the sense defined above. Note that $c_{\text{crit}}(0) = c_{\text{FKPP}}$ in all three cases, since (1) is obtained in the limit as $\varepsilon \to 0$ in (5). The following theorem is the principal result of this article:

Theorem 1.1. *For any reaction-diffusion equation of the form* (5)*, where φ satisfies Assumption $\mathcal{A}$, there exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0)$, the (generalized) critical wave speed $c_{\text{crit}}(\varepsilon)$ for* (5) *is given by*

$$c_{\text{crit}}(\varepsilon) = 2 - \frac{\pi^2}{(\ln \varepsilon)^2} + \mathcal{O}\big((\ln \varepsilon)^{-3}\big). \tag{6}$$

The proof of Theorem 1.1 is carried out in terms of the traveling wave variables $U(\xi) = u(x,t)$ and $\xi = x - ct$. The traveling wave equation corresponding to (5) is given by

$$U'' + cU' + U(1-U^2)\varphi(U, \varepsilon, \tfrac{U}{\varepsilon}) = 0, \tag{7}$$

where the prime denotes differentiation with respect to ξ. For the following analysis, it is convenient to rewrite (7) as a first-order system,

$$\begin{aligned} U' &= V, \\ V' &= -U(1-U^2)\varphi(U, \varepsilon, \tfrac{U}{\varepsilon}) - cV. \end{aligned} \tag{8}$$

Then, traveling wave solutions of (5) correspond to heteroclinic trajectories in (8) which connect the two rest states $U = 1$ and $U = 0$, with

$$\lim_{\xi \to -\infty} (U, V)(\xi) = (1, 0) \quad \text{and} \quad \lim_{\xi \to \infty} (U, V)(\xi) = (0, 0).$$

We will denote the points $(1, 0)$ and $(0, 0)$ in (U, V)-space by Q^- and Q^+, respectively.

Geometrically speaking, the selection of the generalized critical wave speed $c_{\mathrm{crit}}(\varepsilon)$ in (5) is determined by a global condition, given by the requirement that there be a singular heteroclinic orbit Γ for $\varepsilon = 0$ in (8) which perturbs, for $\varepsilon > 0$ small, to a "critical" heteroclinic connection between Q^- and Q^+. More precisely, the desired connection will be established in the intersection of the unstable manifold $\mathcal{W}^u(Q^-)$ of Q^- with the stable manifold $\mathcal{W}^s(Q^+)$ of Q^+ (in cases (i) and (ii)), respectively with the strong stable manifold $\mathcal{W}^{ss}(Q^+)$ of Q^+ (in case (iii)). Consequently, the associated traveling wave solution of (5) will decay to the zero rest state at the strongest possible rate, in accordance with our generalized notion of criticality.

To put it differently, the required persistence of Γ will provide a necessary and sufficient condition which can be applied to uniquely determine $c_{\mathrm{crit}}(\varepsilon)$. This contrasts with the fact that the critical wave speed $c_{\mathrm{FKPP}} = 2$ in the continuum limit (1) is determined locally, see [2, 8, 4, 20, 24, 29]. In the corresponding traveling wave ODE, the origin is a stable node for $c > 2$, a degenerate stable node at $c = 2$, and a stable spiral when $c < 2$. Hence, $U \sim C\xi \mathrm{e}^{-\xi}$ as $\xi \to \infty$ when $c = 2$, whereas the decay is strictly exponential for $c > 2$ and non-monotone for $c < 2$. Therefore, the critical speed $c_{\mathrm{FKPP}} = 2$ is determined by a local transition at the origin. In the cut-off system (8), the traveling wave with critical speed $c_{\mathrm{crit}}(\varepsilon)$ still decays exponentially and monotonically, even though $c_{\mathrm{crit}}(\varepsilon) < 2$.

The main tool used in the proof of Theorem 1.1 is the blow-up technique, also known as geometric desingularization of families of vector fields. This approach is naturally suggested by the fact that the origin is a degenerate fixed point in (8) which can be desingularized via blow-up. In particular, it will allow us to define the (strong) stable manifold of Q^+ in a precise manner. To the best of our knowledge, the blow-up technique was first used in studying limit cycles near a cuspidal loop in [18]. The method has since successfully been applied, including in [16] as an extension of the more classical geometric singular perturbation theory to problems in which normal hyperbolicity is lost; see also [13, 17, 14, 25, 26, 30, 15] and the references therein.

Our proof was guided in part by the results of Section IV in [9]. To relate their analysis to ours, we briefly review their argument after stating the proof of Theorem 1.1.

Moreover, it is worth noting that Brunet and Derrida [9, Section IV] also give physical arguments to show that the asymptotics in (4) are not restricted to their FKPP equation with cut-off in (2), but that the correction to the critical wave speed due to a cut-off in more general equations of similar type will also be $\mathcal{O}((\ln \varepsilon)^{-2})$. This type of behavior should arise in numerical studies of the corresponding discrete models. At the end of this article, we show that Theorem 1.1 generalizes to the cut-off FKPP equation with quadratic nonlinearity,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)\varphi(u, \varepsilon, \tfrac{u}{\varepsilon}), \tag{9}$$

as well as to the larger class of reaction-diffusion equations with cut-off given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \big(u - g(u)\big)\varphi(u, \varepsilon, \tfrac{u}{\varepsilon}), \tag{10}$$

where g satisfies certain properties that will be specified in detail in Section 4.

The effects of cut-offs are also investigated in Section 7 of [31], as one aspect of a broad study of fronts propagating into unstable states. It is shown there that the results of Brunet and Derrida on the shift in propagation speed hold more generally for a large class of so-called fluctuating pulled fronts in the large-N limit. Also, we note that for $\varepsilon = \frac{1}{N} > 0$, these fronts are actually pushed fronts. We refer the reader to [31, Equation (246)], and more generally to Section 7.1 of [31], for further analysis.

Finally, our work complements the results in [28], where a variational principle has been developed to study the shift in the front speed due to cut-off in (9) as well as in more general classes of equations. In particular, lower bounds were obtained for the critical wave speed, and a wide range of trial functions was explored, giving good agreement with the numerics. We think that our results will help to identify the most suitable class of trial functions for this variational approach. Recently, corresponding bounds for a stochastically perturbed FKPP-type equation have been derived in [10], see also the references therein. We hope that the approach developed here will prove useful within that stochastic context, as well.

This article is organized as follows. In Section 2, we carry out the geometric desingularization (blow-up) of the degenerate equilibrium at Q^+, and we construct the singular heteroclinic orbit Γ connecting Q^- and Q^+. Then, in Section 3, we prove that there exists a "critical" heteroclinic solution that lies near this singular heteroclinic, establishing Theorem 1.1. In Section 4, we generalize Theorem 1.1 to the class of reaction-diffusion equations with cut-off in (9) and (10).

2. Blow-up and the singular heteroclinic orbit for (8)

In this section, we construct the singular heteroclinic orbit Γ, i.e., a connection from Q^- to Q^+ for $\varepsilon = 0$. Since $c_{\text{crit}}(\varepsilon) \to 2$ in the singular limit as $\varepsilon \to 0$, this orbit will exist for $c = 2$ in (8). Consequently, we will be concerned with $c = 2$ throughout most of this section. We will only make an exception in Lemma 2.1 and in equations (13), (15), (20), and (22) below, which are stated for general c, in view of their use in Section 3.

The construction of the singular heteroclinic orbit Γ is carried out in the vector field obtained from (8) by desingularizing the origin via the blow-up transformation

$$U = \bar{r}\bar{u}, \quad V = \bar{r}\bar{v}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon}. \tag{11}$$

Here, $(\bar{u}, \bar{v}, \bar{\varepsilon}) \in \mathbb{S}^2 = \big\{(\bar{u}, \bar{v}, \bar{\varepsilon}) \,\big|\, \bar{u}^2 + \bar{v}^2 + \bar{\varepsilon}^2 = 1\big\}$, and $\bar{r} \in [0, r_0]$ for $r_0 > 0$ sufficiently small. In the blow-up process, the degenerate equilibrium at the origin is transformed into the two-sphere $\mathbb{S}^2$. Moreover, since we are interested in $\varepsilon \geq 0$, we only need to consider the half-sphere $\mathbb{S}^2_+$ defined by restricting $\mathbb{S}^2$ to $\bar{\varepsilon} \geq 0$.

The analysis of the induced vector field on $\mathbb{S}^2_+$ is naturally performed in the following two charts: The "classical" rescaling chart K_2 ($\bar{\varepsilon} = 1$), which is used to study the dynamics of (8) in the regime $\{U \leq \varepsilon\}$, as well as one phase-directional chart K_1 ($\bar{u} = 1$), which is employed in the analysis of the regime $\{U \geq \varepsilon\}$. The dynamics in charts K_2 and K_1 are presented in Sections 2.1 and 2.2, respectively. These dynamics are then combined in Section 2.3 to complete the construction of Γ.

The orbit Γ will provide the "backbone" of the perturbative analysis that will be presented in Section 3. More precisely, we will show that, for $\varepsilon > 0$ sufficiently small and c chosen appropriately, a heteroclinic connection will persist close to Γ. Since heteroclinic orbits in (8) correspond to traveling wave solutions of (5), this will prove the existence of such solutions for ε small and an appropriate "critical" value $c_{\rm crit}(\varepsilon)$ of c near $c_{\rm crit}(0) = 2$.

Finally, we note that the analysis in Section 3 will rely heavily on the same two charts defined above: Chart K_2 will be used in proving the existence and uniqueness of $c_{\rm crit}(\varepsilon)$ in Section 3.1, while chart K_1 will play an essential role in Section 3.2, in that it will allow us to prove Theorem 1.1, in a unified manner, for the wide range of cut-off functions that satisfy Assumption $\mathcal{A}$.

***Remark* 2.** For any object $\square$ in the original (U, V, ε)-variables, we will denote the corresponding blown-up object by $\overline{\square}$; in charts K_i $(i = 1, 2)$, the same object will appear as $\square_i$ when necessary.

2.1. **Dynamics in the rescaling chart K_2.** In this subsection, we study system (8) in the regime $\{U \le \varepsilon\}$. This analysis is carried out in the rescaling chart K_2, defined by $\bar{\varepsilon} = 1$ in (11), where the blow-up transformation is given by

$$U = r_2 u_2, \quad V = r_2 v_2, \quad \text{and} \quad \varepsilon = r_2. \tag{12}$$

In terms of these new variables, system (8) becomes

$$u_2' = v_2, \tag{13a}$$

$$v_2' = -u_2(1 - r_2^2 u_2^2)\varphi(r_2 u_2, r_2, u_2) - c v_2, \tag{13b}$$

$$r_2' = 0. \tag{13c}$$

Moreover, since $U < \varepsilon$ if and only if $u_2 < 1$, the cut-off function $\varphi(r_2 u_2, r_2, u_2)$ in K_2 satisfies

$$\varphi(r_2 u_2, r_2, u_2) = \psi(r_2 u_2, r_2, u_2) \quad \text{if } u_2 < 1, \tag{14}$$

by Assumption $\mathcal{A}$. Due to the fact that we are interested in (13) for $U \le \varepsilon$, we extend φ locally in chart K_2 in a continuous manner to $u_2 = 1$ by defining $\varphi(r_2, r_2, 1) = \lim_{u_2 \to 1^-} \psi(r_2 u_2, r_2, u_2)$. Consequently, in $\{u_2 \le 1\}$, the continuous extension of (13) is given by

$$u_2' = v_2, \tag{15a}$$

$$v_2' = -u_2(1 - r_2^2 u_2^2)\psi(r_2 u_2, r_2, u_2) - c v_2, \tag{15b}$$

$$r_2' = 0; \tag{15c}$$

in other words, the values of the orbits of (15) on $\{u_2 = 1\}$ are defined as the limiting values of the corresponding orbits, in $\{u_2 < 1\}$, of (13) as $u_2 \to 1^-$.

For r_0 small, let ℓ_2^+ denote the line of equilibria for system (15) given by

$$\ell_2^+ = \big\{(0, 0, r_2) \,\big|\, r_2 \in [0, r_0]\big\}.$$

For each $r_2 = \varepsilon$ fixed, the associated point on ℓ_2^+ corresponds to the point Q^+ before blow-up. Of most interest to us is the point $(0, 0, 0)$ on ℓ_2^+, which is obtained in the singular limit of $r_2 = 0$. We will denote it by Q_2^+ in the following. A direct calculation reveals

Lemma 2.1. *The point Q_2^+ is semi-hyperbolic for* (15)*, with eigenvalues $\lambda_\pm = \frac{1}{2}(-c \pm \sqrt{c^2 - 4\Psi(0)})$ and* 0*. The corresponding eigenspaces are spanned by $(1, \lambda_\pm, 0)^T$ and $(0, 0, 1)^T$, respectively.*

In particular, it follows that for any $c \sim 2$ in (15), the point Q_2^+ will have one (strong) stable eigendirection associated to the eigenvalue λ_-, and, hence, a one-dimensional stable manifold if $\Psi(0) = 0$, respectively a one-dimensional strong stable manifold in case $\Psi(0) > 0$. The remainder of this section is devoted to the identification of this manifold in the singular limit of $r_2 = 0$, since that is the central object of interest for the dynamics in chart K_2.

As a preliminary step, we introduce the following section,

$$\Sigma_2^{\rm in} = \big\{(1, v_2, r_2) \,\big|\, (v_2, r_2) \in [-v_0, 0] \times [0, r_0]\big\},$$

where $v_0 > 2$ is an appropriately defined constant. This section Σ_2^{in} corresponds to the hyperplane $\{U = \varepsilon\}$, before blow-up, and is an entry face through which orbits of (15) enter the regime $\{0 \leq u_2 \leq 1\}$; see Figure 1. Note that $\{u_2 = 1\}$ separates the regime $\{u_2 > 1\}$, where the dynamics of (15) are unperturbed, from the vertical strip $\{0 \leq u_2 < 1\}$, in which the dynamics are governed by the simplified, cut-off system; cf. Assumption $\mathcal{A}$. Note also that the flow of (15) is directed from the former into the latter, see again Figure 1.

For the remainder of this subsection, we set $c = 2$ in (15), which is due to the fact that $c_{\text{crit}}(0) = 2$ in the singular limit of $r_2 = 0$. When $r_2 = 0$, the resulting system restricted to $\{0 \leq u_2 \leq 1\}$ reduces to

$$
\begin{aligned}
u_2' &= v_2, \\
v_2' &= -u_2\Psi(u_2) - 2v_2.
\end{aligned}
\tag{16}
$$

Moreover, Ψ is $\mathcal{C}^k$-smooth with $k \geq 1$, and there holds $\Psi(u_2) \in [0, 1]$ for all $u_2 \in [0, 1]$ as well as $\Psi(0) \in [0, 1)$, cf. Assumption $\mathcal{A}$.

For the sake of exposition, we first explicitly identify the (strong) stable manifold in case $\Psi \equiv 0$, which corresponds to the Heaviside cut-off analyzed in detail in [9], cf. (ii) above. The more generic case when $\Psi \not\equiv 0$ is treated in Proposition 2.2 below.

With $\Psi \equiv 0$ and $c = 2$, we have $\lambda_+ = 0$ and $\lambda_- = -2$ in Lemma 2.1, i.e., the point Q_2^+ has one stable eigendirection $(1, -2, 0)^T$. The corresponding singular equations in (16) simplify to

$$
\begin{aligned}
u_2' &= v_2, \\
v_2' &= -2v_2,
\end{aligned}
\tag{17}
$$

or, equivalently, to $\frac{dv_2}{du_2} = -2$. The unique solution of this equation with $v_2(0) = 0$ is given by

$$
\Gamma_2 : \; v_2(u_2) = -2u_2. \tag{18}
$$

This invariant straight line is precisely the stable manifold $\mathcal{W}_2^s(Q_2^+)$ of Q_2^+. We also label it by Γ_2, and we will use the two notations interchangeably in the following, since $\mathcal{W}_2^s(Q_2^+)$ will constitute the portion of the singular orbit $\overline{\Gamma}$ in K_2, as we will see shortly. Here, we note that Γ_2 is the unique orbit that is asymptotic to Q_2^+ in K_2 when $\Psi \equiv 0$. Moreover, we define the point $P_2^{\text{in}} = \Gamma_2 \cap \Sigma_2^{\text{in}}$ as the intersection of Γ_2 with the section Σ_2^{in}; more precisely, $P_2^{\text{in}} = (1, -2, 0)$ by (18).

Finally, for $r_2 \in [0, r_0]$ with r_0 sufficiently small, system (15) is a regular ($\mathcal{C}^k$-smooth) perturbation of (17) by Assumption $\mathcal{A}$. Hence, one can define the stable manifold $\mathcal{W}_2^s(\ell_2^+)$ for the entire line ℓ_2^+. Note that $\mathcal{W}_2^s(\ell_2^+)$ will correspond to the stable manifold $\mathcal{W}^s(Q^+)$ of Q^+ after blow-down. Note also that the regularity of this manifold depends on the choice of φ, i.e., $\mathcal{W}_2^s(\ell_2^+)$ is in general only $\mathcal{C}^k$-smooth, with k again defined as in Assumption $\mathcal{A}$. Moreover, for any r_2 small enough and constant, the corresponding leaf of $W_2^s(\ell_2^+)$ is a small, $\mathcal{C}^k$ perturbation of Γ_2. The geometry in chart K_2 is illustrated in Figure 1.

In case $\Psi \not\equiv 0$, the analysis proceeds in a similar manner. However, there is an important difference in that one cannot necessarily find an explicit expression for Γ_2. Nevertheless, Γ_2 can still be uniquely defined by using the following result:

Proposition 2.2. *There exists a $\mathcal{C}^k$-smooth function $\gamma_2 : [0, 1] \to \mathbb{R}$ such that the (strong) stable manifold Γ_2 of Q_2^+ in (16) is given by $\Gamma_2 = \{(u_2, v_2) \,|\, u_2 \in [0, 1], v_2 = \gamma_2(u_2)\}$. The graph of γ_2 lies between $\{v_2 = -2u_2\}$ and $\{v_2 = -u_2\}$; there holds in particular $\gamma_2(0) = 0$ and $-2 \leq \gamma_2(1) < -1$.*

Proof. Given $\Psi \not\equiv 0$, we distinguish between the two cases of $\Psi(0) = 0$ and $\Psi(0) > 0$ here, beginning with the former.

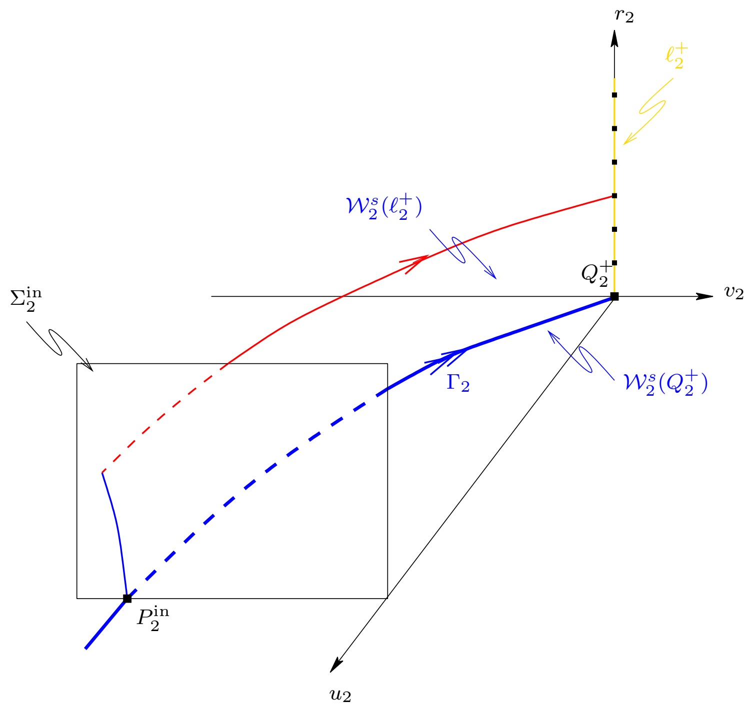

FIGURE 1. The geometry in chart K_2.

If $\Psi(0) = 0$, the origin in (16) is semi-hyperbolic, see Lemma 2.1, with a double zero eigenvalue and a unique, one-dimensional stable manifold that can locally be represented as the graph of a $\mathcal{C}^k$ function γ_2. Moreover, since $-2u_2 \le \gamma_2(u_2)$, there clearly holds $-2u_2 \le \gamma_2(u_2) < -u_2$ for u_2 small.

For $\Psi(0) > 0$, Q_2^+ is a stable node for (16) by Lemma 2.1, with one strong stable eigendirection. To be able to apply standard invariant manifold theory [21], we introduce the new (projective) variable $\tilde{v}_2 = \frac{v_2}{u_2}$ in (16). The resulting system of equations

$$\begin{aligned} u_2' &= u_2\tilde{v}_2, \\ \tilde{v}_2' &= -\Psi(u_2) - 2\tilde{v}_2 - \tilde{v}_2^2 \end{aligned}$$

is again $\mathcal{C}^k$-smooth, with two equilibrium points in $\{u_2 = 0\}$ and the corresponding $\tilde{v}_2$-coordinates given by $\tilde{v}_2 = -1 \pm \sqrt{1 - \Psi(0)}$. We focus on the lower equilibrium $(u_2, \tilde{v}_2) = (0, -1 - \sqrt{1 - \Psi(0)})$, since it corresponds to the direction of the strong stable manifold of Q_2^+ before the transformation.

Linearization shows that this point is a hyperbolic saddle, with a unique stable manifold that is described locally by the graph of some $\mathcal{C}^k$ function $\tilde{\gamma}_2$. We now define the function γ_2 via $\gamma_2(u_2) = u_2\tilde{\gamma}_2(u_2)$. Since $-2 \le -1 - \sqrt{1 - \Psi(0)} < -1$ by Assumption $\mathcal{A}$, it follows that $-2u_2 \le \gamma_2(u_2) < -u_2$ for u_2 small.

To show that these local definitions as well as the corresponding bounds on γ_2 can be extended to $u_2 \in [0, 1]$ in both cases, we note that in the region under consideration, we have the estimate

$$-2 \le \frac{dv_2}{du_2} < -\frac{u_2 + 2v_2}{v_2}$$

on $\frac{dv_2}{du_2}$. (This estimate easily follows by comparing the equations in (16) to the two extreme cases of $\Psi \equiv 1$ and $\Psi \equiv 0$, respectively, and by recalling that $v_2 < 0$.) Since, moreover, $u_2 + v_2 \le 0$, we find $-2 \le \frac{dv_2}{du_2} < -1$, which implies in particular $-2 \le \gamma_2(1) < -1$. This completes the proof. $\square$

Given Proposition 2.2, it follows that the intersection of Γ_2 with $\Sigma_2^{\rm in}$ is now given by a point $P_2^{\rm in} = (1, v_2^{\rm in}, 0)$ with $-2 \leq v_2^{\rm in} < -1$. Finally, we remark that by Assumption $\mathcal{A}$, for each $r_2 \in [0, r_0]$ sufficiently small, the corresponding leaf of the stable manifold $\mathcal{W}_2^s(\ell_2^+)$ (in case $\Psi(0) = 0$), respectively of the strong stable manifold $\mathcal{W}_2^{ss}(\ell_2^+)$ (in case $\Psi(0) > 0$), will still be a regular perturbation of Γ_2. In analogy to the notation introduced for $\Psi \equiv 0$ above, these manifolds will be denoted by $\mathcal{W}^s(Q^+)$, respectively by $\mathcal{W}^{ss}(Q^+)$, after blow-down.

***Remark* 3.** System (17) corresponds exactly to Equation (23) of [9] in their "region III."

***Remark* 4.** If $\Psi(0) = 0$ and if, moreover, 0 is an accumulation point of positive zeros of Ψ, (16) will have non-trivial equilibria on the u_2-axis, in addition to Q_2^+. However, we did not need to consider these equilibria, since only points with $u_2 = 0$ can correspond to Q^+ after blow-down.

2.2. **Dynamics in the phase-directional chart K_1.** In this subsection, we study the dynamics of system (8) in the regime $\{U \geq \varepsilon\}$. To that end, we work in the directional chart K_1, which is defined by $\bar{u} = 1$, and in which the blow-up transformation reads

$$U = r_1, \quad V = r_1 v_1, \quad \text{and} \quad \varepsilon = r_1 \varepsilon_1. \tag{19}$$

Also, to relate the analyses of this and the previous subsections, we will use the following relationship between the variables in (12) and (19) on the domain of overlap between charts K_1 and K_2:

Lemma 2.3. *The change of coordinates $\kappa_{12} : K_1 \to K_2$ is given by*

$$u_2 = \frac{1}{\varepsilon_1}, \quad v_2 = \frac{v_1}{\varepsilon_1}, \quad \textit{and} \quad r_2 = r_1 \varepsilon_1.$$

For the inverse change $\kappa_{21} = \kappa_{12}^{-1} : K_2 \to K_1$, there holds

$$r_1 = r_2 u_2, \quad v_1 = \frac{v_2}{u_2}, \quad \textit{and} \quad \varepsilon_1 = \frac{1}{u_2}.$$

Here, we note that both κ_{12} and κ_{21} are well-defined as long as ε_1 and u_2, respectively, are finite and bounded away from zero. Correspondingly, the overlap domain between K_1 and K_2 includes $\{U = \varepsilon\}$, where $\varepsilon_1 = 1$, respectively $u_2 = 1$. This fact will enable us to connect the dynamics in the two charts there, see Section 2.3 below.

In terms of the variables in (19), system (8) becomes

$$r_1' = r_1 v_1, \tag{20a}$$

$$v_1' = -(1 - r_1^2)\varphi(r_1, r_1\varepsilon_1, \tfrac{1}{\varepsilon_1}) - cv_1 - v_1^2, \tag{20b}$$

$$\varepsilon_1' = -\varepsilon_1 v_1. \tag{20c}$$

Since, moreover, chart K_1 is used to analyze (8) in the regime $\{U \geq \varepsilon\}$ and since $U > \varepsilon$ if and only if $1 > \varepsilon_1$, it follows that φ satisfies

$$\varphi(r_1, r_1\varepsilon_1, \tfrac{1}{\varepsilon_1}) \equiv 1 \quad \text{if } 1 > \varepsilon_1, \tag{21}$$

by Assumption $\mathcal{A}$. To extend φ in chart K_1 in a continuous manner to $\varepsilon_1 = 1$, we define $\varphi(r_1, r_1, 1) = 1$ in (20). This will allow us to continue the dynamics of (20) in $\{\varepsilon_1 < 1\}$ up to $\varepsilon_1 = 1$; more precisely, in $\{\varepsilon_1 \leq 1\}$, the continuous extension of system (20) reduces to

$$r_1' = r_1 v_1, \tag{22a}$$

$$v_1' = -(1 - r_1^2) - cv_1 - v_1^2, \tag{22b}$$

$$\varepsilon_1' = -\varepsilon_1 v_1. \tag{22c}$$

Hence, given an orbit of (22) in $\{\varepsilon_1 \leq 1\}$, its value on $\{\varepsilon_1 = 1\}$ can be regarded as the limiting value for $\varepsilon_1 \to 1^-$ of the corresponding orbit, in $\{\varepsilon_1 < 1\}$, of (20).

The equilibria of (22) are found as follows: For $v_1 \neq 0$, we have to examine (22) with $r_1 = 0$ and $\varepsilon_1 = 0$, which implies $\varepsilon = r_1\varepsilon_1 = 0$. Therefore, $c = 2$ in (22b), and the only equilibrium is located at $P_1 = (0, -1, 0)$ in that case. Other equilibria are obtained for $v_1 = 0$ in (22); these lie on the line $\ell_1^- = \{(1, 0, \varepsilon) \,|\, \varepsilon \in [0, \varepsilon_0]\}$, which corresponds to the original point Q^- before blow-up. In particular, for $\varepsilon = 0$, we will denote the point $(1, 0, 0)$ on ℓ_1^- by Q_1^-.

We focus on P_1 here; a straightforward calculation shows

Lemma 2.4. *The eigenvalues of* (22) *linearized at* P_1 *are given by* -1, 0, *and* 1, *with eigenvectors* $(1, 0, 0)^T$, $(0, 1, 0)^T$, *and* $(0, 0, 1)^T$, *respectively.*

The planes $\{\varepsilon_1 = 0\}$ and $\{r_1 = 0\}$ are invariant for (22). To identify the portion of the singular orbit $\overline{\Gamma}$ lying in K_1, we analyze the dynamics of (22) separately in these two invariant planes.

The first portion of Γ_1, which we label Γ_1^-, is forward asymptotic to P_1 and lies in $\{\varepsilon_1 = 0\}$. Since the governing equations in this invariant plane are equivalent to the original (unmodified) FKPP equation (1), Γ_1^- corresponds precisely to the unstable manifold $\mathcal{W}^u(Q^-)$ of Q^- for $\varepsilon = 0$ in (8) or, equivalently, to the "tail" of the FKPP heteroclinic orbit after blow-up. The situation is summarized in Figure 2.

For the subsequent analysis, we require the following key fact on the asymptotics of Γ_1^- in chart K_1. Define a new section Σ_1^{in} by

$$\Sigma_1^{\text{in}} = \{(r_0, v_1, \varepsilon_1) \,|\, (v_1, \varepsilon_1) \in [-v_0, 0] \times [0, 1]\},$$

where the constant v_0 has the same value as in Section 2.1, and let $P_1^{\text{in}} = \Gamma_1^- \cap \Sigma_1^{\text{in}}$ denote the intersection of Γ_1^- with Σ_1^{in}, i.e., $P_1^{\text{in}} = (r_0, v_1^{\text{in}}, 0)$; see Figure 2.

Lemma 2.5. *The orbit* Γ_1^- *is tangent to* $(0, 1, 0)^T$ *(i.e., to the* v_1*-axis) as* $\Gamma_1^- \to P_1$.

Proof. The assertion follows from a straightforward phase plane argument. Consider the original first-order system (8), and recall that it corresponds to the unmodified FKPP equation (1) in the singular limit of $\varepsilon = 0$. Moreover, recall that for $c = 2$, the point Q^+ is a degenerate node, with a unique, $\mathcal{C}^\infty$-smooth, one-dimensional, invariant manifold that corresponds precisely to the strong stable manifold $\mathcal{W}^{ss}(Q^+)$ in this case. Note that $\mathcal{W}^{ss}(Q^+)$ agrees with the stable manifold of P_1 after transformation to chart K_1, and let $P_1^s = (r_0, v_1^s, 0)$ denote the point of intersection of this manifold with Σ_1^{in}. Expanding $\mathcal{W}^{ss}(Q^+)$ about Q^+, one finds $V(U) = -U - \frac{1}{2}U^3 + \mathcal{O}(U^5)$ and, hence, $v_1^s < -1$.

To see where Γ_1^- will lie with respect to the manifold $\mathcal{W}^{ss}(Q^+)$ after blow-up, we construct a trapping region for the flow of (8). On the U-axis given by $\{V = 0\}$, there holds

$$\begin{aligned} U' &= 0, \\ V' &= -U(1 - U^2), \end{aligned}$$

and, hence, $(0, 1) \cdot (0, -U(1 - U^2))^T = -U(1 - U^2) < 0$, since $U < 1$. Similarly, on $\{V = -U\}$,

$$\begin{aligned} U' &= -U, \\ V' &= U(1 + U^2), \end{aligned}$$

and therefore $(1, 1) \cdot (-U, U(1 + U^2))^T = U^3 > 0$. Thus, the flow of (8) is trapped in the wedge bounded by the lines $\{V = 0\}$ and $\{V = -U\}$.

In particular, for the flow of the blown-up vector field (22) in chart K_1, this implies that the singular orbit Γ_1^- corresponding to $\mathcal{W}^u(Q^-)$ must enter the equivalent of that trapping region in K_1, i.e., it must intersect Σ_1^{in} in a point $P_1^{\text{in}} = (r_0, v_1^{\text{in}}, 0)$ with $v_1^{\text{in}} > v_1^s$. Therefore, it follows that Γ_1^- is tangent to the v_1-axis as $\Gamma_1^- \to P_1$. $\square$

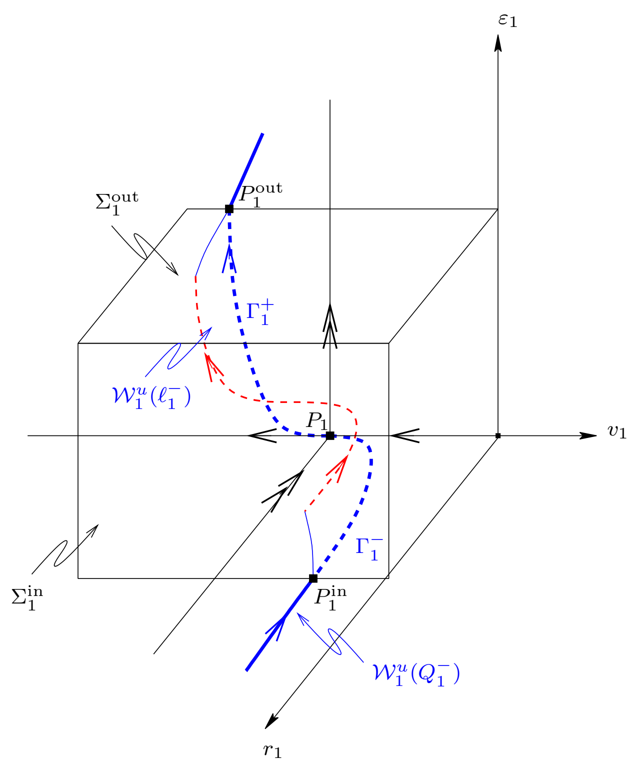

FIGURE 2. The geometry in chart K_1.

The second portion of the singular orbit Γ_1 is backward asymptotic to P_1 and lies in the invariant plane $\{r_1 = 0\}$; it is labeled Γ_1^+ in Figure 2. In $\{r_1 = 0\}$, system (22) is given by

$$\begin{aligned} v_1' &= -(1+v_1)^2, \\ \varepsilon_1' &= -\varepsilon_1 v_1, \end{aligned} \tag{23}$$

where we have again simplified the right-hand side of (22b) using the fact that $c = 2$ for $r_1 = 0$. Rewriting (23) with ε_1 as the independent variable, we find

$$\frac{dv_1}{d\varepsilon_1} = \frac{(1+v_1)^2}{\varepsilon_1 v_1}.$$

This equation is separable and can be solved explicitly; the explicit solution is

$$v_1(\varepsilon_1) = -\frac{1 + \mathrm{W}(\frac{\alpha}{\varepsilon_1})}{\mathrm{W}(\frac{\alpha}{\varepsilon_1})}. \tag{24}$$

Here, the Lambert W-function is defined as the solution of

$$\mathrm{W}(\mathrm{z}) \cdot \mathrm{e}^{\mathrm{W}(\mathrm{z})} = \mathrm{z},$$

where α is a real constant and W denotes the principal branch of the Lambert W-function.

To complete the construction of Γ_1^+, we need to determine the value of α in (24). Fix a section Σ_1^{out} via

$$\Sigma_1^{\text{out}} = \big\{(r_1, v_1, 1) \,\big|\, (r_1, v_1) \in [0, r_0] \times [-v_0, 0]\big\};$$

here, r_0 and v_0 are defined as before. Hence, under the coordinate transformation κ_{12} between the variables in charts K_1 and K_2 (recall Lemma 2.3), this new section coincides with the section $\Sigma_2^{\rm in}$; that is, $\kappa_{12}(\Sigma_1^{\rm out}) = \Sigma_2^{\rm in}$. Consequently, it again corresponds to $\{U = \varepsilon\}$, before blow-up.

More importantly, the entry point $P_2^{\rm in}$ of the singular orbit Γ_2 in chart K_2 determines a unique point in $\Sigma_1^{\rm out}$ that allows us to uniquely fix α in (24). Let $P_1^{\rm out} = \kappa_{21}(P_2^{\rm in})$, and note that $P_1^{\rm out} = (0, v_1^{\rm out}, 1)$, see Lemma 2.3 and the definition of $P_2^{\rm in}$. Hence, it remains to find α such that $v_1(1) = v_1^{\rm out}$. We will analyze the two cases of $\Psi \equiv 0$ and $\Psi \not\equiv 0$ separately, beginning with the former.

In case $\Psi \equiv 0$, we have $P_1^{\rm out} = (0, -2, 1)$, where the v_1-coordinate $v_1^{\rm out}$ of $P_1^{\rm out}$ is determined via $v_1^{\rm out} = \frac{v_2^{\rm in}}{u_2^{\rm in}} = -2$. Therefore, $v_1(1) = -2$, which implies that $\alpha = {\rm e}$ and

$$\Gamma_1^+ :\ v_1(\varepsilon_1) = -\frac{1 + {\rm W}(\frac{\rm e}{\varepsilon_1})}{{\rm W}(\frac{\rm e}{\varepsilon_1})} \tag{25}$$

in $\{r_1 = 0\}$. Finally, by Taylor expanding (25) for ε_1 small, we find that the orbit Γ_1^+ is tangent to the v_1-axis as $\Gamma_1^+ \to P_1$.

In the more generic case when $\Psi \not\equiv 0$, the orbit Γ_1^+ can be constructed, and the corresponding value of α found, in a similar fashion. In particular, note that the geometry in K_1 is the same for all functions φ that satisfy Assumption $\mathcal{A}$. Hence, one only needs to require that $\mathcal{W}_2^s(Q_2^+)$ can be extended (in K_2) to the section $\Sigma_2^{\rm in}$, in backward "time." In fact, recalling the definition of $P_2^{\rm in}(= \Gamma_2 \cap \Sigma_2^{\rm in}) = (1, v_2^{\rm in}, 0)$, one finds $-2 \le v_2^{\rm in} < -1$ by Proposition 2.2. Also, $v_1 \to -1$ as $\varepsilon_1 \to 0$ in (24) regardless of the value of α, i.e., $\Gamma_1^+ \to P_1$ in K_1 tangent to the v_1-axis. Therefore, to complete the construction of Γ_1^+ it only remains to show that α can still be fixed such that $v_1^{\rm out} = v_2^{\rm in}$. Now, a direct computation reveals that

$$\alpha = -\frac{1}{v_1^{\rm out} + 1} {\rm e}^{-\frac{1}{v_1^{\rm out}+1}} \quad \text{for } v_1^{\rm out} < -1 \quad \text{and} \quad \alpha = 0 \quad \text{for } v_1^{\rm out} = -1. \tag{26}$$

Summarizing the above calculations, we have established

Lemma 2.6. *The orbit Γ_1^+ is tangent to the v_1-axis as $\Gamma_1^+ \to P_1$.*

The geometry in chart K_1 is illustrated in Figure 2.

***Remark* 5.** Equations (23) correspond precisely to Equation (22) of [9] in their "region II."

***Remark* 6.** In this subsection, we analyzed the dynamics in the regime $\{U \ge \varepsilon\}$ using chart K_1. The dynamics in this regime could also have been studied in chart K_2. In fact, in $\{u_2 \ge 1\}$, the continuous extension of system (13) in K_2 yields

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -u_2(1 - r_2^2 u_2^2) - c v_2, \end{aligned} \tag{27}$$

which of course again corresponds to the unmodified FKPP equation after blow-up. Now, recall that in the singular limit of $r_2 = 0$, we have $c = 2$ in (27). Therefore, for $r_2 = 0$, system (27) further simplifies and is equivalent to Equation (22) of [9].

2.3. **Connecting the dynamics in charts K_2 and K_1.** In this subsection, we prove the existence of a singular heteroclinic connection from Q_1^- to Q_2^+:

Proposition 2.7. *Let φ be a cut-off function which satisfies Assumption $\mathcal{A}$. Then, there exists a singular heteroclinic orbit $\overline{\Gamma}$ for equations* (15) *and* (22)*, respectively, that connects Q_1^- to Q_2^+.*

An illustration can be found in Figure 3.

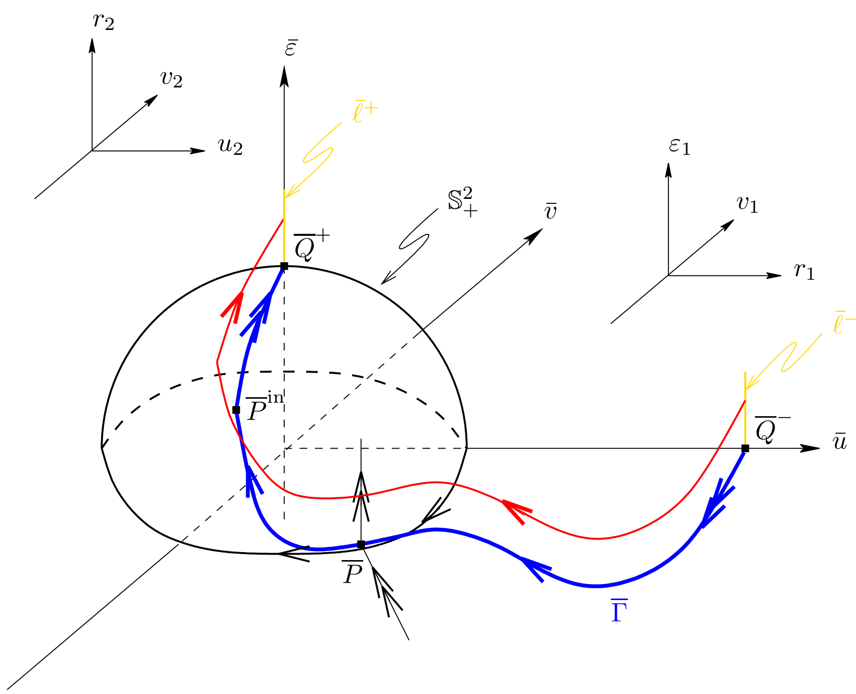

FIGURE 3. The global geometry of the blown-up vector field.

Proof. To construct $\overline{\Gamma}$, we combine the results obtained in charts K_2 and K_1 in the previous subsections. The connection between the dynamics in the individual charts is established in $\{\bar{u} = \bar{\varepsilon}\}$, which corresponds to Σ_2^{in} in K_2, respectively to Σ_1^{out} in K_1. For the sake of exposition, we restrict ourselves to the case when $\Psi \equiv 0$ in the following; the argument in case $\Psi \not\equiv 0$ is completely analogous.

Recall that on the blown-up locus given by $\{\bar{r} = 0\}$, there holds $r_2 = 0$, respectively $r_1 = 0$; hence, systems (15) and (22), respectively, reduce to (17) and (23). Moreover, recall the expressions (18) and (25), obtained for Γ_2 and Γ_1^+, respectively. Given the definitions of the points P_2^{in} and P_1^{out} above, one sees immediately that Γ_2 and Γ_1^+ connect in a $\mathcal{C}^0$ manner in $\Sigma_2^{\text{in}} = \kappa_{12}(\Sigma_1^{\text{out}})$. This shows the existence of a singular connection from P_1 to Q_2^+.

For the connection between Q_1^- and P_1, recall that in the invariant plane $\{\varepsilon_1 = 0\}$, system (22) corresponds precisely to the original FKPP equation (1). Hence, the heteroclinic orbit connecting Q_1^- and P_1 is given by the corresponding FKPP orbit, denoted by Γ_1^- above, after blow-up.

In sum, the desired singular heteroclinic connection $\overline{\Gamma}$ is thus given by the union of the orbits Γ_1^-, Γ_1^+, and Γ_2 and of the singularities Q_1^-, P_1, and Q_2^+, see Figure 3. This completes the proof. $\square$

The orbit $\overline{\Gamma}$ is unique if $\Psi(0) = 0$ and if, in addition, 0 is an accumulation point of positive zeros of Ψ (as is the case, for example, when $\Psi \equiv 0$), since Γ_2 is the unique orbit in K_2 which is asymptotic to Q_2^+ then and since the corresponding value of α required in the definition of Γ_1^+ is also unique, cf. (26). This verifies the claim made in (ii) above, namely, that there is a unique value of c in this case for which traveling wave solutions to (5) exist.

In case Ψ has an isolated zero at 0, or when $\Psi(0) > 0$, there will be an infinity of orbits asymptoting into Q_2^+ in K_2. However, Γ_2 can again be uniquely defined, respectively, as the stable manifold of Q_2^+ when 0 is an isolated zero of Ψ and as the strong stable manifold when $\Psi(0) > 0$. In both cases, it is that particular orbit for which the decay rate is the strongest, cf. Proposition 2.2.

All other orbits which asymptote into Q_2^+ do so tangent to, respectively, a center manifold (in the former case) and to the weak stable eigendirection of Q_2^+ (in the latter case); therefore, the associated decay rates must be weaker. This justifies the assertions made in (i) and (iii), since $\overline{\Gamma}$ then corresponds, to leading order, to the traveling wave solution in (5) for which the decay at the origin is strongest, in accordance with the generalized definition of a "critical" wave speed in that case.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We start by showing that, for each $\varepsilon > 0$ sufficiently small, there exists a unique value $c_{\text{crit}}(\varepsilon)$ of c in (8), $c_{\text{crit}}(\varepsilon) \sim 2$, for which there is a heteroclinic orbit connecting Q^- to Q^+ that lies in the intersection of $\mathcal{W}^u(Q^-)$ with $\mathcal{W}^s(Q^+)$, respectively with $\mathcal{W}^{ss}(Q^+)$, and that is close to the singular orbit Γ constructed in the previous section. Then, we derive corresponding necessary conditions involving the dependence of $c_{\text{crit}}(\varepsilon)$ on ε in order for Γ to persist. Combining these two aspects of the analysis, we obtain the existence of a unique (generalized) critical speed $c_{\text{crit}}(\varepsilon)$ as well as the leading-order expansion of $c_{\text{crit}}(\varepsilon)$, as stated in Theorem 1.1. We emphasize that the invariant manifolds $\mathcal{W}^u(Q^-)$, $\mathcal{W}^s(Q^+)$, and $\mathcal{W}^{ss}(Q^+)$ depend on the system parameters in (8), including c; however, as is customary in dynamical systems theory, that dependence may be suppressed in the notation.

The required analysis is summarized in the following subsections. In Section 3.1, we demonstrate the existence of a "critical" heteroclinic connection for a unique $c = c_{\text{crit}}(\varepsilon)$ in (8), with $\varepsilon > 0$ sufficiently small, using the original variables (U, V) in combination with the "classical" rescaling chart K_2. In Section 3.2, the transition from the unperturbed into the modified, cut-off regime in (8) is naturally described in chart K_1, which also provides a clear geometric understanding of Theorem 1.1.

3.1. **Existence and uniqueness of $c_{\text{crit}}(\varepsilon)$.** We set out by proving that, for $\varepsilon > 0$ in (8), the unstable manifold of Q^- intersects the (strong) stable manifold of Q^+ for a unique value of c, labeled $c_{\text{crit}}(\varepsilon)$. In particular, we show that $c_{\text{crit}}(\varepsilon) < 2$:

Proposition 3.1. *For $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small and $c \sim 2$, there exists a unique $c_{\text{crit}}(\varepsilon)$ such that for $c = c_{\text{crit}}(\varepsilon)$ in (8), there is a "critical" heteroclinic orbit connecting Q^- and Q^+. Moreover, there holds $c_{\text{crit}}(\varepsilon) < 2$.*

Proof. The proof is given for general φ and, consequently, for any Ψ that is admissible by Assumption $\mathcal{A}$.

Recall the definition of the section Σ_2^{in} in chart K_2, as well as of the point $P_2^{\text{in}} = (1, v_2^{\text{in}}, 0)$. For $r_2 (= \varepsilon)$ sufficiently small, the intersection of the stable manifold $\mathcal{W}^s(\ell_2^+)$ (respectively of the strong stable manifold $\mathcal{W}_2^{ss}(\ell_2^+)$) with Σ_2^{in} can be written as the graph of a $\mathcal{C}^k$ function $v_2^{\text{in}} = v_2^{\text{in}}(c, \varepsilon)$. Here, k is as in the definition of φ, see Assumption $\mathcal{A}$. In the singular limit of $\varepsilon = 0$, it follows from Proposition 2.2 that $-2 \leq v_2^{\text{in}}(2, 0) < -1$ for any admissible Ψ. Moreover, given that for general, fixed c, (16) becomes

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -u_2 \Psi(u_2) - c v_2, \end{aligned}$$

one easily sees that $\frac{\partial v_2^{\text{in}}}{\partial c}(2, 0) < 0$. (Note that in the special case when $\Psi \equiv 0$, one may explicitly find these quantities as $v_2^{\text{in}}(2, 0) = -2$ and $\frac{\partial v_2^{\text{in}}}{\partial c}(2, 0) = -1$.) Hence, $\frac{\partial v_2^{\text{in}}}{\partial c}(c, \varepsilon) < 0$ for $c \sim 2$ and $\varepsilon > 0$ small enough, and it follows by regular perturbation theory that the intersection of the (strong) stable manifold of Q^+ with $\{U = \varepsilon\}$, which is given by $V^{\text{in}}(c, \varepsilon) \equiv \varepsilon v_2^{\text{in}}(c, \varepsilon)$ after blow-down, certainly satisfies $-3\varepsilon < V^{\text{in}}(c, \varepsilon) < -\varepsilon$ as well as $\frac{\partial V^{\text{in}}}{\partial c}(c, \varepsilon) < 0$.

To describe the unstable manifold $\mathcal{W}^u(Q^-)$ of Q^- on $\{U \geq \varepsilon\}$, with $\varepsilon > 0$, we consider the equations in (8) for $\varphi \equiv 1$:

$$\begin{aligned} U' &= V, \\ V' &= -U(1-U^2) - cV. \end{aligned} \tag{28}$$

The intersection of $\mathcal{W}^u(Q^-)$ with $\{U = \varepsilon\}$ can be represented as the graph of an analytic function $V^{\text{out}} = V^{\text{out}}(c, \varepsilon)$, where $\frac{\partial V^{\text{out}}}{\partial c} > 0$.

Now, for any $c \lesssim 2$ fixed, a standard phase plane argument shows that the limit as $\varepsilon \to 0$ in $V^{\text{out}}(c, \varepsilon)$, which represents $\mathcal{W}^u(Q^-) \cap \{U = 0\}$, is well-defined, as well as that $V^{\text{out}}(c, 0) < 0$. Hence, for $\varepsilon > 0$ small enough, $V^{\text{out}}(c, \varepsilon)$ must also be strictly $\mathcal{O}(1)$ and negative, which, together with $V^{\text{in}}(c, \varepsilon) > -3\varepsilon$, implies that $V^{\text{in}} > V^{\text{out}}$ for $c \lesssim 2$.

It remains to consider the case where $c = 2$, with $\varepsilon > 0$ small: Recall that by the proof of Lemma 2.5, the flow of (28) is trapped in the wedge bounded by the lines $\{V = 0\}$ and $\{V = -U\}$, which shows that in $\{U = \varepsilon\}$, $V^{\text{out}}(2, \varepsilon) \geq -\varepsilon$ for $\varepsilon > 0$ sufficiently small. Since $V^{\text{in}}(2, \varepsilon) < -\varepsilon$, as before, it follows that $V^{\text{in}} < V^{\text{out}}$ for $c = 2$.

In sum, we conclude that $\mathcal{W}^s(Q^+)$ and $\mathcal{W}^u(Q^-)$, respectively $\mathcal{W}^{ss}(Q^+)$ and $\mathcal{W}^u(Q^-)$, must connect to each other in $\{U = \varepsilon\}$ for some value of c, which we call $c_{\text{crit}}(\varepsilon)$; moreover, we note that by the above argument, $c_{\text{crit}}(\varepsilon) < 2$. Finally, since $\frac{\partial V^{\text{in}}}{\partial c} < 0$ and $\frac{\partial V^{\text{out}}}{\partial c} > 0$ for $c \sim 2$ and $\varepsilon > 0$ small, it follows that $c_{\text{crit}}(\varepsilon)$ is unique. This completes the proof. $\square$

3.2. **Transition through chart K_1.** To study the passage of trajectories through chart K_1 under the flow of (22), it is again convenient to work with appropriate sections for the flow. We will employ the sections $\Sigma_1^{\text{out}} = \kappa_{21}(\Sigma_2^{\text{in}})$ and Σ_1^{in} defined above. The transition of orbits through chart K_1 from Σ_1^{in} to Σ_1^{out} is governed by the transition map $\Pi_1 : \Sigma_1^{\text{in}} \to \Sigma_1^{\text{out}}$. Our aim is to derive a sufficiently accurate approximation of this map.

Taking into account that $c_{\text{crit}}(\varepsilon) \to 2$ in the singular limit as $\varepsilon \to 0$, we first define $\tilde{c}(\varepsilon) = c_{\text{crit}}(\varepsilon) - 2$, where we note that $\tilde{c}(\varepsilon) = o(1)$ for $\varepsilon \to 0$. Since $\tilde{c}(\varepsilon)$ must be strictly negative for the heteroclinic connection whose existence was demonstrated in Proposition 3.1, we set $\tilde{c}(\varepsilon) = -\eta(\varepsilon)^2$. Moreover, we shift the point $P_1 = (0, -1, 0)$ to the origin by introducing the new variable $w = v_1 + 1$. As before, let P_1^{in} denote the point of intersection of Γ_1^- with Σ_1^{in}. Since $\tilde{c} \lesssim 0$, we restrict ourselves to describing Π_1 on $\Sigma_1^{\text{in}} \cap \{v_1 < v_1^{\text{in}}\}$ in the following.

Under the above transformations, the equations in (22) become

$$r_1' = -r_1(1-w), \tag{29a}$$

$$w' = r_1^2 - w^2 - \eta^2(1-w), \tag{29b}$$

$$\varepsilon_1' = \varepsilon_1(1-w). \tag{29c}$$

In turn, after a rescaling of time by a division through the positive factor $1-w$, we find that system (29) can be written as

$$\dot{r}_1 = -r_1, \tag{30a}$$

$$\dot{w} = -\eta^2 + \frac{r_1^2 - w^2}{1-w}, \tag{30b}$$

$$\dot{\varepsilon}_1 = \varepsilon_1. \tag{30c}$$

(Here, the overdot denotes differentiation with respect to the new rescaled time ξ_1.) Plainly, the ε_1-equation (30c) decouples, and we have $\varepsilon_1(\xi_1) = \frac{\varepsilon}{r_0}\mathrm{e}^{\xi_1}$. To study the (r_1, w)-subsystem in (30), we change notation, replacing r_1 by s. Moreover, we introduce η as a third variable. Hence, in the following we analyze the system of equations

$$\dot{s} = -s, \tag{31a}$$

$$\dot{w} = -\eta^2 + \frac{s^2 - w^2}{1 - w}, \tag{31b}$$

$$\dot{\eta} = 0. \tag{31c}$$

Remark 7. Before giving a rigorous analysis of the transition past P_1 and the proof of Theorem 1.1, we present a heuristic argument. Consider the leading-order approximation to (31b),

$$\dot{w} = -\eta^2 + s^2 - w^2 + \mathcal{O}(3) \sim -\eta^2 - w^2,$$

and note that the transition "time" from Σ_1^{in} to Σ_1^{out} under the flow of (30) is given by $\Xi_1 = -\ln \frac{\varepsilon}{r_0}$. Hence, by separation of variables, one finds that to leading order,

$$-\frac{1}{\eta} \arctan\Big(\frac{w}{\eta}\Big)\Big|_{w^{\mathrm{in}}}^{w^{\mathrm{out}}} = -\ln \frac{\varepsilon}{r_0}.$$

Here, $w^{\mathrm{in}} \approx 1 + v_1^{\mathrm{in}}$ and $w^{\mathrm{out}} \approx 1 + v_1^{\mathrm{out}}$ are assumed to be "almost independent" of ε. Evaluating the arctangent and taking into account that $\eta = \mathcal{o}(1)$ by assumption, one obtains $\frac{1}{\eta}(-\pi) \sim \ln \varepsilon$. Solving for η to obtain $\eta \sim -\frac{\pi}{\ln \varepsilon}$ and recalling that $\tilde{c} = -\eta^2$, one retrieves the leading-order correction to the critical wave speed stated in Theorem 1.1.

To analyze rigorously the transition through chart K_1 in the vicinity of the point P_1 (which is now located at the origin) for $\varepsilon \in (0, \varepsilon_0)$ small, we have to describe the map Π_1 in more detail. More precisely, the fact that the unstable manifold $\mathcal{W}^u(Q^-)$ connects to the stable manifold $\mathcal{W}^s(Q^+)$ (respectively to the strong stable manifold $\mathcal{W}^{ss}(Q^+)$), i.e., that $\mathcal{W}_1^u(\ell_1^-)$ is "matched" to $\mathcal{W}_2^s(\ell_2^+)$ (respectively to $\mathcal{W}_2^{ss}(\ell_2^+)$) in $\Sigma_2^{\mathrm{in}} = \kappa_{12}(\Sigma_1^{\mathrm{out}})$ after transition past P_1, imposes a particular dependence of η on ε.

To leading order, an expression for $\eta(\varepsilon)$ is derived in the following proposition:

Proposition 3.2. *For a "critical" heteroclinic connection between Q^- and Q^+ to be possible when $\varepsilon > 0$ in* (8)*, there must necessarily hold*

$$\eta(\varepsilon) = -\frac{\pi}{\ln \varepsilon} + \mathcal{O}\big((\ln \varepsilon)^{-2}\big). \tag{32}$$

Proof. To simplify the analysis of (31), we make a normal form transformation which decouples the dynamics of s and w in (31). By Theorem 1 of [5], there exists, for each $r \geq 1$, a $\mathcal{C}^r$ coordinate change

$$(s, w, \eta) \mapsto \big(S(s, w, \eta), W(s, w, \eta), \eta\big) \tag{33}$$

with $S(0, w, \eta) = 0$ which transforms (31) into

$$\dot{S} = -S, \tag{34a}$$

$$\dot{W} = -\eta^2 - \frac{W^2}{1 - W}, \tag{34b}$$

$$\dot{\eta} = 0. \tag{34c}$$

Note that (33) respects the invariance of $\{s = 0\}$ and additionally leaves the level surfaces $\{\eta = \eta_0\}$ (with η_0 constant) invariant.

For η sufficiently small, we need to calculate the transition "time" $\widetilde{\Xi}_1$ of solutions of system (34) between the two sections corresponding to Σ_1^{in} and Σ_1^{out} after transformation by (33). Let $W^{\mathrm{in}} > 0$ and $W^{\mathrm{out}} < 0$ denote the corresponding values of W. We will see that, to leading order, $\widetilde{\Xi}_1 = \widetilde{\Xi}_1(W^{\mathrm{in}}, W^{\mathrm{out}}, \eta)$ is independent of the exact values of W^{in} and W^{out}.

Since the equations in (34) are decoupled, we can solve (34b) by separation of variables. To that end, we introduce a new variable $Z = W - \frac{\eta^2}{2}$ in (34b), which gives

$$-d\tilde{\xi}_1 = \frac{(1 - Z - \frac{\eta^2}{2})\, dZ}{Z^2 + \eta^2(1 - \frac{\eta^2}{4})}.$$

Integrating, we find

$$-\widetilde{\Xi}_1 = \frac{1-\frac{\eta^2}{2}}{\eta\sqrt{1-\frac{\eta^2}{4}}} \arctan\Big(\frac{Z}{\eta\sqrt{1-\frac{\eta^2}{4}}}\Big)\Big|_{Z^{\rm in}}^{Z^{\rm out}} - \frac{1}{2}\ln\big|Z^2 + \eta^2\big(1 - \tfrac{\eta^2}{4}\big)\big|\Big|_{Z^{\rm in}}^{Z^{\rm out}}. \tag{35}$$

Here, $Z^{\rm in}$ and $Z^{\rm out}$ are the values of Z obtained from $W^{\rm in}$ and $W^{\rm out}$, respectively.

Reverting to W in (35) and dividing out a factor of η^{-1}, we find

$$\begin{aligned} -\widetilde{\Xi}_1 = \frac{1}{\eta}\Bigg[\frac{1-\frac{\eta^2}{2}}{\sqrt{1-\frac{\eta^2}{4}}}\Bigg[\arctan\Big(\frac{W^{\rm out}-\frac{\eta^2}{2}}{\eta\sqrt{1-\frac{\eta^2}{4}}}\Big) - \arctan\Big(\frac{W^{\rm in}-\frac{\eta^2}{2}}{\eta\sqrt{1-\frac{\eta^2}{4}}}\Big)\Bigg] \\ - \frac{\eta}{2}\Big[\ln\big|(W^{\rm out})^2 - W^{\rm out}\eta^2 + \eta^2\big| - \ln\big|(W^{\rm in})^2 - W^{\rm in}\eta^2 + \eta^2\big|\Big]\Bigg]. \end{aligned} \tag{36}$$

Since we are only interested in deriving a leading-order expression for η, we expand $(1 - \frac{\eta^2}{2})(1 - \frac{\eta^2}{4})^{-\frac{1}{2}} = 1 + \mathcal{O}(\eta^2)$. Also, since $w^{\rm out} < 0$ and $w^{\rm in} > 0$ and since (33) is near-identity, we conclude that $W^{\rm out} < 0$ and $W^{\rm in} > 0$ are $\mathcal{O}(1)$ as $\varepsilon \to 0$ and independent of η to leading order.

To derive expansions for the arctangent-terms in (36), we make use of the identity

$$\arctan(x) + \arctan\Big(\frac{1}{x}\Big) = \pm\frac{\pi}{2},$$

where the sign equals the sign of x. In particular, for $|x|$ large, we have

$$\arctan(x) = \pm\frac{\pi}{2} - \Big(\frac{1}{x} - \frac{1}{3x^3} + \dots\Big).$$

In our case,

$$x = \frac{W-\frac{\eta^2}{2}}{\eta\sqrt{1-\frac{\eta^2}{4}}} = \frac{W}{\eta}\big(1 + \mathcal{O}(\eta^2)\big)$$

and, hence,

$$\arctan\Big(\frac{W^{\rm out}-\frac{\eta^2}{2}}{\eta\sqrt{1-\frac{\eta^2}{4}}}\Big) = -\frac{\pi}{2} - \frac{\eta}{W^{\rm out}} + \mathcal{O}(\eta^3) \quad \text{and} \quad \arctan\Big(\frac{W^{\rm in}-\frac{\eta^2}{2}}{\eta\sqrt{1-\frac{\eta^2}{4}}}\Big) = \frac{\pi}{2} - \frac{\eta}{W^{\rm in}} + \mathcal{O}(\eta^3).$$

To estimate the logarithmic terms in (36), we note that $\ln|W^2 - W\eta^2 + \eta^2| = \ln|W^2| + \ln|1 - \frac{\eta^2}{W^2}(1 - W)| = 2\ln|W| + \mathcal{O}(\eta^2)$. Collecting these estimates, we find

$$\widetilde{\Xi}_1 = \frac{1}{\eta}\Big[\pi + \eta\Big[\frac{1}{W^{\rm out}} - \frac{1}{W^{\rm in}} + \ln\Big|\frac{W^{\rm out}}{W^{\rm in}}\Big|\Big] + \mathcal{O}(\eta^3)\Big]. \tag{37}$$

On the other hand, we know that S evolves according to $S = S^{\rm in}{\rm e}^{-\tilde{\xi}_1}$, where $S^{\rm in} > 0$ denotes the initial value $S(0)$. Hence, we may also write

$$\tilde{\xi}_1 = -\ln\frac{S}{S^{\rm in}} = -\ln\big(s\beta(s, w, \eta)\big), \tag{38}$$

where $\beta(s, w, \eta)$ is a strictly positive, $\mathcal{C}^r$-smooth function that depends on the choice of normalizing coordinates in (33). (Note that the ε-dependence of β is implicitly encoded in its arguments s, w, and η, and that $\beta = \mathcal{O}(1)$ as $\varepsilon \to 0$.) Now, during that same "time" $\widetilde{\Xi}_1$ introduced above, the S-variable has to change from $S^{\rm in}$ to a value $S^{\rm out}$ permitting a connection between $\mathcal{W}_1^u(\ell_1^-)$ and

$\mathcal{W}_2^s(\ell_2^+)$, respectively $\mathcal{W}_2^{ss}(\ell_2^+)$, in $\Sigma_2^{\rm in} = \kappa_{12}(\Sigma_1^{\rm out})$. This will impose a relation between η and ε that can be described, to leading order, in the normal form coordinates (S, W, η).

To that end, recall that s in (31) stands for r_1, as defined in chart K_1, as well as that r_1 in $\Sigma_1^{\rm out}$ is fixed to $r_1^{\rm out} = \varepsilon$. Hence, we obtain from (38) that

$$\tilde{\tilde{\Xi}}_1 = -\ln\left(\varepsilon\tilde{\beta}(\varepsilon,\eta)\right) \tag{39}$$

for some function $\tilde{\beta}(\varepsilon,\eta)$ which is strictly positive and $\mathcal{C}^r$-smooth, with $\tilde{\beta} = \mathcal{O}(1)$ for $\varepsilon \to 0$.

Combining (37) and (39) and recalling that $W^{\rm in}$ and $W^{\rm out}$ are $\mathcal{O}(1)$ as $\varepsilon \to 0$, as noted below equation (36), we find

$$-\ln\varepsilon = \frac{1}{\eta}\left[\pi + \eta\theta(\varepsilon) + \mathcal{O}(\eta^2)\right] \tag{40}$$

for some bounded function θ. (In fact, we can assume that θ is $\mathcal{C}^{\min\{k,r\}}$-smooth, see Assumption $\mathcal{A}$ and (33).) Solving (40) for η, we obtain

$$\eta = -\frac{\pi}{\ln\varepsilon} + \tilde{\eta}, \tag{41}$$

where $\tilde{\eta}$ defines a relative correction in (41), i.e., there holds $\tilde{\eta} = o((\ln\varepsilon)^{-1})$. In fact, substituting (41) into (37), one can check that $\tilde{\eta} = \mathcal{O}((\ln\varepsilon)^{-2})$: Given (40), it follows that

$$\tilde{\eta}(\ln\varepsilon)^2 = (\pi - \tilde{\eta}\ln\varepsilon)\theta(\varepsilon) + \mathcal{O}\big((\ln\varepsilon)^{-1}, \tilde{\eta}, \tilde{\eta}^2\ln\varepsilon\big).$$

Since $\tilde{\eta}\ln\varepsilon = o(1)$ by assumption, we have $\tilde{\eta} = \mathcal{O}((\ln\varepsilon)^{-2})$. This concludes the proof. $\square$

Now, the assertions of Theorem 1.1 follow immediately from Propositions 3.1 and 3.2. By Proposition 3.1, given $\varepsilon > 0$ sufficiently small, there exists a "critical" heteroclinic connection in (8) close to Γ for a unique value $c_{\rm crit}(\varepsilon)$ of c. This connecting orbit lies in the intersection of the two manifolds $\mathcal{W}^u(Q^-)$ and $\mathcal{W}^s(Q^+)$ (in cases (i) and (ii)), respectively $\mathcal{W}^u(Q^-)$ and $\mathcal{W}^{ss}(Q^+)$ (in case (iii)), and corresponds, by construction, to the traveling wave solution of (5) with the strongest possible decay at the zero rest state, in accordance with our generalized notion of criticality. In addition, by Proposition 3.2, $c_{\rm crit}(\varepsilon)$ must necessarily satisfy

$$c_{\rm crit}(\varepsilon) = 2 + \tilde{c}(\varepsilon) = 2 - \eta(\varepsilon)^2 = 2 - \frac{\pi^2}{(\ln\varepsilon)^2} + \mathcal{O}\big((\ln\varepsilon)^{-3}\big).$$

This completes the proof of Theorem 1.1.

***Remark* 8.** While it suffices to take $k = 1$ in Assumption $\mathcal{A}$ to establish Theorem 1.1, the computation of higher-order terms in the expansion of $c_{\rm crit}(\varepsilon)$ will generally require stronger regularity assumptions on φ.

***Remark* 9.** An alternative way of analyzing (31) would be to make use of the fact that $\mathcal{S}_0 : \{s = 0\}$ is an invariant manifold which is normally attracting and which has a fast, strong stable foliation. Hence, the dynamics in chart K_1 can be decomposed into the dynamics along the fast fibers and the slow motion of the associated base points on $\mathcal{S}_0$ [19].

Alternatively still, instead of proving Proposition 3.2 via an explicit calculation, one could apply the blow-up technique again inside $\mathcal{S}_0$ to obtain an approximation for the transition map Π_1.

***Remark* 10.** As stated in Section 1, our proof of Theorem 1.1 was guided in part by the results of Section IV in [9]. These were derived by dividing the phase space of the traveling wave ODE (7) into three regions, an inner "region III" about the origin ($U < \varepsilon$) in which $\varphi \equiv 0$, an outer "region I" ($U = \mathcal{O}(1)$) in which $\varphi \equiv 1$, and an intermediate "region II" ($\varepsilon < U \ll 1$) in which φ makes the transition from zero to one. Then, asymptotic matching was used at the interfaces between these regions. To relate our analysis to that of [9], we briefly summarize their argument here. In region I, equation (5) is precisely the original FKPP equation. Hence, the asymptotic form of the corresponding solution to (7) is $U_{\mathrm{I}}(\xi) \sim A\xi \mathrm{e}^{-\xi}$ for ξ large. In the intermediate region II, equation (7) reduces to leading order to the linear equation $U'' + cU' + U = 0$, see Remark 5. The asymptotic form of the solution for large ξ is given by $U_{\mathrm{II}}(\xi) \sim C\mathrm{e}^{-\gamma_r \xi} \sin(\gamma_i \xi)$, where $\gamma_r \pm \gamma_i$ are the roots of $\lambda^2 - c\lambda + 1$, and we note that $\gamma_r - 1 = \mathcal{O}(c - c_{\mathrm{FKPP}})$ and $\gamma_i = \mathcal{O}(\sqrt{c - c_{\mathrm{FKPP}}})$. Matching U_{I} and U_{II} to leading order in $\sqrt{c - c_{\mathrm{FKPP}}}$ implies that $C = \frac{A}{\gamma_i}$. Finally, in the inner region III, equation (7) reduces to $U'' + cU' = 0$, and the solution satisfies $U_{\mathrm{III}}(\xi) \sim \varepsilon \mathrm{e}^{-c(\xi - \xi_0)}$, where $\xi = \xi_0$ when $U = \varepsilon$, see also Remark 3. The requirement of continuity of U and U' at the interface between regions II and III then implies (after some calculation) that $\gamma_i \xi_0 \sim \pi$ and $\xi_0 \sim -\ln \varepsilon$. Therefore, one directly obtains $\gamma_i \sim \frac{\pi}{|\ln \varepsilon|}$, and, hence, that $c - c_{\mathrm{FKPP}} \sim -\frac{\pi^2}{(\ln \varepsilon)^2}$ as $\varepsilon \to 0$.

4. Generalization of Theorem 1.1 to (9) and (10)

In this section, we generalize the result of Theorem 1.1 to reaction-diffusion equations with cut-off other than (5), focusing in particular on equations (9) and (10). In fact, it suffices to consider equation (10) only, since (9) is a special case of (10) with $g(u) = u^2$.

Thus, we are concerned with the general class of reaction-diffusion equations with cut-off in (10), where $g(u)$ is chosen such that

(i) $g(u) = \mathcal{O}(u^2)$ as $u \to 0$,
(ii) there exists a $q^- > 0$ such that $g(q^-) = q^-$,
(iii) $g'(q^-) > 1$, and
(iv) $0 < g(u) < u$ for all $u \in (0, q^-)$.

Note that the function g does not have to be analytic or even $\mathcal{C}^\infty$-smooth; a finite degree of differentiability suffices. Geometrically, the above conditions may be interpreted as follows. Condition (i) guarantees that $Q^+ = (0, 0)$ is an equilibrium of the corresponding traveling wave ODE $U'' + cU' + (U - g(U))\varphi(U, \varepsilon, \frac{U}{\varepsilon}) = 0$, cf. (7), and that it is again a degenerate stable node for $c = 2$, with the same linearization as before. Conditions (ii) and (iii) guarantee that $Q^- = (q^-, 0)$ is a hyperbolic saddle equilibrium of that same traveling wave ODE. Finally, condition (iv) implies that there are no equilibria apart from Q^+ and Q^- and that the trapping region constructed in Lemma 2.5 exists also for these more general equations.

Given conditions (i)–(iv), the analysis of the previous two sections carries over almost verbatim. In particular, the origin is still a degenerate equilibrium for (10) which can again be desingularized via the blow-up transformation in (11). The analysis can again be performed in the same two charts K_2 and K_1, with the dynamics in chart K_2 being exactly as given in Section 2.1. In chart K_1, the only modification that has to be made is that the r_1^2-term in equation (20b) is now replaced by $\frac{g(r_1)}{r_1}$. The rest of the analysis, however, proceeds as above.

In particular, one obtains precisely the same normal form system (34) as in Section 3. Therefore, the result of Theorem 1.1 also holds exactly for the more general class of equations in (10), and the first-order correction to the continuum wave speed is again given by $-\frac{\pi^2}{(\ln \varepsilon)^2}$.

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