Magnetic Force Formulae for Magnets at Small Distances

Nikola Popović [1], **Dirk Praetorius** [2], and **Anja Schlömerkemper** [*3]

[1] Boston University, Department of Mathematics and Statistics and Center for BioDynamics, 111 Cummington Street, Boston, MA 02215, U.S.A.

[2] Vienna University of Technology, Institute for Analysis and Scientific Computing, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria

[3] Max Planck Institute for Mathematics in the Sciences, Inselstraße 22-26, D-04103 Leipzig, Germany

In [7], the magnetic force on subregions of rigid magnetized bodies was studied as a discrete-to-continuum limit. The derived force formula includes a new term, which depends on the underlying crystalline lattice structure $\mathcal{L}$. It originates from contributions of magnetic dipole-dipole interactions of dipole moments close to the boundary of the considered subregion.

Further studies of this new term have led to the question of how the magnetic force between two idealized magnets, which are a distance $\varepsilon > 0$ apart, depends on ε as $\varepsilon \to 0$. In this article, analytical aspects of this question are discussed, cf. [5], where also numerical experiments are performed.

1 Introduction

Ferromagnetic shape memory alloys have some potential as new micro-devices, cf. e.g. [1]. In order to construct such devices, a better fundamental understanding of the dynamics of moving interfaces is of interest, cf. e.g. [3] for the study of a micro-scale cantilever. In this context, the question arises which mathematical formula describes the force between two parts of a magneto-elastic material best. There is a long list of related literature, cf. the references in [7].

To begin with, we assume the magnetized material to be rigid. Several formulae are known for the magnetic force that is exerted by one subbody on another [6, 7]. While one can bring the formulae into a form such that the volume force densities are the same, the surface force densities are different. Therefore, there is some need for experiments to clarify which formula is the most appropriate. Unfortunately, it is not possible or at least not obvious how to measure magnetic forces in the interior of a magnetic body. To circumvent this difficulty, we suggest to consider the force between two magnetic bodies – instead of looking at two subregions of one magnetic body. In particular, we discuss the force between two polygonal bodies A and B whose boundaries have at least a set of positive surface measure in common. Even for this case, one finds several different force formulae.

In this article, we outline some analytical results from [5], to which we refer for details. For the sake of brevity, we focus on polygonal three-dimensional permanent magnets A and B with finitely many edges such that $A \cap B = \emptyset$ and the surface measure of $\partial A \cap \partial B$ is positive. Let $\mathbf{m}_A : A \to \mathbb{R}^3$ denote the magnetization corresponding to A; we assume $\mathbf{m}_A$ to be a Lipschitz-continuous vector field which is trivially extended to the entire space $\mathbb{R}^3$, i.e., $\mathbf{m}_A = \mathbf{m}_A \chi_A$. Let $\mathbf{H}_A$ be the magnetic field which is generated by $\mathbf{m}_A$ and which is obtained from the magnetostatic Maxwell equations $\mathrm{curl}\, \mathbf{H}_A = 0$ and $\mathrm{div}\, (\mathbf{H}_A + \mathbf{m}_A) = 0$ (in some appropriate physical units). We adopt the same notation for B and denote with $\mathbf{H}_{A \cup B}$ the magnetic field generated by $\mathbf{m} := \mathbf{m}_A + \mathbf{m}_B$.

The experimental idea which drives both the analytical and the numerical studies in [5] is the following: Take A and B to be a distance $\varepsilon > 0$ apart and measure the magnetic force between them as ε gets smaller. Then, compare the experimental results for $\varepsilon \to 0$ with the various mathematical formulae under discussion. To prepare and motivate such real-life experiments, we derive several magnetic force formulae within the geometric framework given above and perform the corresponding numerical experiments.

2 Classical force formula for separated bodies

Following the above ideas, we first assume A and B to be a distance $\varepsilon > 0$ apart; for definiteness, we shift B along the $(1, 0, 0)$-axis and define $B_\varepsilon = \{x + \varepsilon(1, 0, 0) \,|\, x \in B\}$. Then, we apply the classical, well-accepted magnetic force formula for separated bodies (cf. e.g. [4]),

$$\mathbf{F}^{(\mathrm{sep},\varepsilon)} = \int_A (\mathbf{m}_A \cdot \nabla)\mathbf{H}_{B_\varepsilon}\, dV.$$

Several numerical experiments involving this formula are performed in [5]. There, we also prove that the limit $\mathbf{F}^{(\mathrm{sep},0)}$ for $\varepsilon \to 0$ exists and that for any $\varepsilon \geq 0$, the force $\mathbf{F}^{(\mathrm{sep},\varepsilon)}$ can be computed via closed-form formulae.

[*] Corresponding author: e-mail: schloem@mis.mpg.de, Phone: +49 341 9959 547, Fax: +49 341 9959 633

3 Formula derived from a discrete setting of magnetic dipoles

Secondly, we consider two magnets A and B in contact, i.e., we take the surface measure of $\partial A \cap \partial B$ to be positive. We focus on a magnetic force formula which is obtained from a discrete setting of magnetic dipole moments, cf. [7]. Here, we present a generalization of the theorem to polygonal domains which are in contact, but not necessarily nested. In [5], we additionally consider another formula $\mathbf{F}^{(\mathrm{Brown})}$, which was intensively studied in [2] and mathematically analyzed in [6].

Let $\mathcal{L}$ be a Bravais lattice describing the crystalline structure of the material, e.g. $\mathcal{L} = \mathbb{Z}^3$. For each $x \in \frac{1}{\ell}\mathcal{L}$, $\ell \in \mathbb{N}$, we introduce a magnetic dipole moment $\mathbf{m}^{(\ell)}(x) = \frac{1}{\ell^3}\,\mathbf{m}(x)$ and denote its i-th component by $\mathbf{m}_i^{(\ell)}(x)$. The magnetic force between the dipole moments in $A \cap \frac{1}{\ell}\mathcal{L}$ and those in $B \cap \frac{1}{\ell}\mathcal{L}$ is given by superposition of all dipole-dipole interactions [7, 5]:

$$\mathbf{F}^{(\ell)} = \frac{1}{4\pi} \sum_{i,j=1}^{3} \sum_{x \in \bar{A} \cap \frac{1}{\ell}\mathcal{L}} \mathbf{m}_i^{(\ell)}(x) \sum_{y \in B \cap \frac{1}{\ell}\mathcal{L}} \nabla \partial_i \partial_j |x-y|^{-1}\, \mathbf{m}_j^{(\ell)}(y).$$

Theorem 3.1 ([5]**)** *Under the above assumptions on A, B, $\mathbf{m}_A$ and $\mathbf{m}_B$, the limit $\lim_{\ell\to\infty} \mathbf{F}^{(\ell)} =: \mathbf{F}^{(\mathrm{lim})}$ exists and*

$$\begin{aligned} \mathbf{F}_k^{(\mathrm{lim})} = {} & \int_A (\mathbf{m}_A \cdot \nabla)(\mathbf{H}_{A\cup B})_k \, dV + \frac{1}{2} \int_{\partial A} \big((\mathbf{m}_A - \mathbf{m}_B)\cdot \mathbf{n}_A\big)(\mathbf{m}_A \cdot \mathbf{n}_A)(\mathbf{n}_A)_k \, d\Gamma \\ & + \frac{1}{2} \sum_{i,j,p=1}^{3} S_{ijkp} \int_{\partial A \cap \partial B} (\mathbf{m}_A)_i (\mathbf{m}_B)_j (\mathbf{n}_A)_p \, d\Gamma, \end{aligned} \tag{1}$$

where $\mathbf{n}_A$ denotes the outer normal to ∂A and

$$S_{ijkp} := -\frac{1}{4\pi} \lim_{\delta\to 0} \lim_{\ell\to\infty} \sum_{z \in B_\delta \cap \frac{1}{\ell}\mathcal{L}\setminus\{0\}} \big(\partial_k \partial_i \partial_j (\varphi^{(\delta)}(z)|z|^{-1})\big)\, z_p \, \frac{1}{\ell^3} \tag{2}$$

for an arbitrary smooth function $\varphi^{(\delta)} : \mathbb{R}^3 \to [0,1]$ with $\varphi(z) = 1$ if $|z| < \frac{\delta}{2}$ and $\varphi(z) = 0$ if $|z| > \delta$.

The first two terms in (1) are also of interest by themselves; we therefore set these terms equal to $\mathbf{F}^{(\mathrm{long})}$ and refer to this formula as the long range force. It turns out that $\mathbf{F}^{(\mathrm{sep},0)} = \mathbf{F}^{(\mathrm{long})}$. The third term in (1), called the short range force $\mathbf{F}^{(\mathrm{short})}$, is an additional surface term, which was first obtained in [6, 7].

4 Discussion and further development

Our analytical and numerical studies [5] are driven by two questions: (i) Which of the force formulae derived within the continuum theory is the "best" one? To that end, we compare $\mathbf{F}^{(\mathrm{Brown})}$ with $\mathbf{F}^{(\mathrm{sep},0)} = \mathbf{F}^{(\mathrm{long})}$. (ii) Is the additional surface force contribution $\mathbf{F}^{(\mathrm{short})}$ of physical relevance? That is, we ask whether the discrete-to-continuum limit $\mathbf{F}^{(\mathrm{lim})}$ is a good formula to describe the magnetic force between two magnets that are in contact.

In the numerical experiments in [5], we consider idealized magnets of cuboidal shape with constant magnetizations at distances $\varepsilon > 0$ and $\varepsilon = 0$, respectively. We make a first basic observation here. To be specific, let A and B be two unit cubes which touch at one face, with normal $\mathbf{n}_A = (1,0,0)$. Moreover, let $\mathbf{m}_A = (1,0,0)$ and $\mathbf{m}_B = (1,0,0)$. Then, $\mathbf{F}_1^{(\mathrm{long})} \approx 5.5$. According to (2), we need to fix some underlying lattice structure to calculate the short range force term. We choose $\mathcal{L} = \mathbb{Z}^3$ and obtain $\mathbf{F}_1^{(\mathrm{short})} = \frac{1}{2}\, S_{1111}\, |\partial A \cap \partial B| \approx 0.67$. Hence, $(\mathbf{F}_1^{(\mathrm{lim})} - \mathbf{F}_1^{(\mathrm{long})})/\mathbf{F}_1^{(\mathrm{long})} = \mathbf{F}_1^{(\mathrm{short})}/\mathbf{F}_1^{(\mathrm{long})} \approx 12.2\%$. Though this estimate is based on a number of idealizations, it indicates that the contribution coming from $\mathbf{F}^{(\mathrm{short})}$ is of some relevance to the total force that one magnet exerts on the other, which would be interesting to verify in real-life experiments.

In Theorem 3.1, the domains A and B are assumed to be in contact. The above observation together with the numerical experiments for $\varepsilon > 0$ in [5] raises the following question: How does the short range contribution $\mathbf{F}^{(\mathrm{short})}$ change if A and B are not in contact, but a small distance $\varepsilon > 0$ apart? This question, however, requires new analytical techniques for the passage from the discrete setting to the continuum.

References

[1] K. Bhattacharya and R. D. James, The material is the machine, Science **307**, 53–54 (2005).
[2] W. F. Brown, Magnetoelastic Interactions, Springer-Verlag, Berlin (1966).
[3] R. D. James, Configurational Forces in Magnetism with Application to the Dynamics of a Small-Scale Ferromagnetic Shape Memory Cantilever, Continuum Mech. Thermodyn. **14**, 55–86 (2002).
[4] L. D. Landau, E. M. Lifschitz and L. P. Pitaevskii, Electrodynamics of Continuous Media, 2nd ed., Pergamon, Oxford (1984).
[5] N. Popović, D. Praetorius and A. Schlömerkemper, Analysis and numerical simulation of magnetic forces between rigid polygonal bodies, Preprint.
[6] A. Schlömerkemper, Magnetic forces in discrete and continuous systems, Doctoral thesis, University of Leipzig (2002)
[7] A. Schlömerkemper, Mathematical derivation of the continuum limit of the magnetic force between two parts of a rigid crystalline material, Arch. Rational Mech. Anal. **176**(2), 227–269 (2005).