A GEOMETRIC ANALYSIS OF THE LAGERSTROM MODEL: EXISTENCE OF SOLUTIONS AND RIGOROUS ASYMPTOTIC EXPANSIONS

NIKOLA POPOVIĆ*

Institute for Analysis
and Scientific Computing,
Vienna University of Technology,
Wiedner Hauptstraße 8-10,
A-1040 Vienna, Austria, EU.
E-mail: nikola.popovic@tuwien.ac.at

We give a geometric singular perturbation analysis of a classical problem proposed by Lagerstrom to illustrate the ideas involved in the rather intricate asymptotic treatment of low Reynolds number flow. We present a geometric proof based on the blow-up method for the existence and uniqueness of solutions. Moreover, we show how asymptotic expansions for these solutions can be obtained in this framework, thereby establishing a connection to the method of matched asymptotic expansions.

1. Lagerstrom's model equation

Lagerstrom's model equation was first introduced to elucidate the ideas and techniques used in the asymptotic treatment of incompressible flow past a solid at low Reynolds number ($n = 2, 3,\ 0 \leq \varepsilon \ll 1,\ \xi \in [1, \infty]$):[2]

$$u'' + \frac{n-1}{\xi}u' + \varepsilon u u' = 0 \tag{1a}$$

$$u(\xi = 1) = 0, \qquad u(\xi = \infty) = 1. \tag{1b}$$

Classically, such problems have been analyzed using the method of *matched asymptotic expansions*;[1,6] here, similar difficulties as in the original problem arise (*Stokes' paradox, Whitehead's paradox*). Our approach, which is based on geometric (*dynamical systems*) methods, gives a novel explanation of these phenomena, leading to a better understanding of the singular perturbation nature of the problem.[3,4,5]

*This work has been supported by the Austrian Science Foundation under grant Y 42-MAT.

2. A dynamical systems approach

Setting $\eta := \xi^{-1} \in [0,1]$, we rewrite (1a),(1b) as

$$\begin{aligned} u' &= v \\ v' &= -(n-1)\eta v - \varepsilon u v \\ \eta' &= -\eta^2 \\ \varepsilon' &= 0 \end{aligned} \tag{2a}$$

$$u(\xi = 1) = 0, \quad \eta(\xi = 1) = 1, \quad u(\xi = \infty) = 1. \tag{2b}$$

There is a line ℓ of non-hyperbolic equilibria for (2a); hence, results from standard invariant manifold theory do not apply directly. To analyze the dynamics near ℓ, we define a *(polar) blow-up transformation* for (2a),(2b):

$$\Phi : \begin{cases} \mathbb{R} \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^4 \\ (\bar{u}, \bar{v}, \bar{\eta}, \bar{\varepsilon}, \bar{r}) \mapsto (\bar{u}, \bar{r}\bar{v}, \bar{r}\bar{\eta}, \bar{r}\bar{\varepsilon}). \end{cases} \tag{3}$$

The resulting vector field is studied by introducing two charts (K_1 and K_2), which correspond to the inner and outer regions in the classical approach.

3. Existence of solutions

To prove existence and (local) uniqueness of solutions to (1a),(1b), we employ a *shooting argument*: we track a manifold $\mathcal{V}$ of admissible inner boundary values and show that it intersects *transversely* the stable manifold $\mathcal{W}^s$ of a point $Q \in \ell$ corresponding to the outer boundary condition.

Theorem 3.1. [3,4] *For $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small and $n = 2, 3$, there exists a locally unique solution to Lagerstrom's model equation (1a),(1b).*

The proof is constructive, and is carried out in the blown-up coordinates. For $n = 2$, the argument is considerably more involved than for $n = 3$. A particular difficulty is the occurrence of *resonances* in chart K_1.

4. Rigorous asymptotic expansions

To leading order, an expansion for $v_\varepsilon := v\big|_{\xi=1} = u'\big|_{\xi=1}$ is given by[2]

$$v_\varepsilon = 1 - \varepsilon \ln \varepsilon - (\gamma + 1)\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{4}$$

for $n = 3$ (a similar result can be obtained for $n = 2$).

Classically, the *logarithmic terms* in (4) have been accounted for under the notion of *switchback*; we show that they are due to resonance. Our approach is rigorous, as our expansions are approximations to well-defined geometric objects, namely, to *invariant manifolds* of (2a).

We begin by deriving expansions in K_2, making an ansatz of the form

$$v_2(u_2, \eta_2) = \sum_{j=0}^{\infty} C_j(\eta_2)(u_2 - 1)^j. \tag{5}$$

Inserting (5) into the corresponding equations in K_2 yields

Proposition 4.1. [3,5] *For $j \geq 1$, $C_j(\eta_2)$ can be written as*

$$C_j(\eta_2) = \eta_2 e^{-\eta_2^{-1}} \sum_{\substack{k,l=0 \\ l \leq k}}^{\infty} \gamma_{kl}^j \eta_2^{-k} (\ln \eta_2)^l. \tag{6}$$

Given Proposition 4.1, in K_1 we expand $v_1(u_1, \varepsilon_1)$ as

$$v_1(u_1, \varepsilon_1) = \sum_{\substack{i,j=0 \\ j \leq i}}^{\infty} a_{ij}(u_1) \varepsilon_1^i (\ln \varepsilon_1)^j. \tag{7}$$

Proposition 4.2. [3,5] *There exist unique smooth functions $a_{ij}(u_1)$ such that (5) and (7), seen as double expansions, are the same.*

Expansion (7), when evaluated at the inner boundary in K_1, gives precisely expansion (4). Due to extensive switchback, the case $n = 2$ is computationally more demanding.

References

1. S. Kaplun and P. A. Lagerstrom, Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. *J. Math. Mech.* **6**, 585-593, 1957.
2. P. A. Lagerstrom, *Matched asymptotic expansions: ideas and techniques.* Volume **76** of *Applied Mathematical Sciences*, Springer-Verlag, New York, 1988.
3. N. Popović, *A Geometric Analysis of the Lagerstrom Model: Existence of Solutions and Rigorous Asymptotic Expansions*, PhD thesis, Vienna University of Technology, 2002.
4. N. Popović and P. Szmolyan, A geometric analysis of the Lagerstrom model problem. *J. Diff. Eqs.* **199**, 290-325, 2004.
5. N. Popović and P. Szmolyan, Rigorous asymptotic expansions for Lagerstrom's model equation—a geometric approach. Submitted.
6. I. Proudman and J. R. A. Pearson, Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* **2**, 237-262, 1957.