A GEOMETRIC ANALYSIS OF THE LAGERSTROM MODEL PROBLEM

NIKOLA POPOVIĆ AND PETER SZMOLYAN

Abstract. Lagerstrom's model problem is a classical singular perturbation problem which was introduced to illustrate the ideas and subtleties involved in the analysis of viscous flow past a solid at low Reynolds number by the method of matched asymptotic expansions. In this paper, the corresponding boundary value problem is analyzed geometrically by using methods from the theory of dynamical systems, in particular invariant manifold theory. As an essential part of the dynamics takes place near a line of non-hyperbolic equilibria, a blow-up transformation is introduced to resolve these singularities. This approach leads to a constructive proof of existence and local uniqueness of solutions and to a better understanding of the singular perturbation nature of the problem. In particular, the source of the logarithmic switchback phenomenon is identified.

1. Introduction

Viscous flow past a solid at low Reynolds number is a classical singular perturbation problem from fluid dynamics. Steady low Reynolds number flow of an incompressible fluid past a circular cylinder was studied by [Sto51]: as a first approximation, he took the Reynolds number to be zero in the governing equations and found that the resulting boundary value problem has no solution (*Stokes paradox*). For flow past a sphere, Stokes did in fact find an approximation which has been widely used. In an attempt to derive a higher-order approximation for the spherical case, however, [Whi89] found that the next term has a singularity at infinity (*Whitehead paradox*). More than half a century later, [Ose10] observed that these seeming paradoxes were due to an incorrect treatment of the flow far from the cylinder respectively the sphere and could be avoided by linearizing about the flow at infinity. Oseen solved the resulting equation for spherical flow and obtained a solution which improves Stokes' solution; however, he failed to give a systematic expansion procedure. The conceptual structure of the problem was clarified much later by Kaplun and Lagerstrom [Kap57, KL57] and Proudman and Pearson [PP57], who showed that it could be solved by the systematic use of the method of *matched asymptotic expansions.*

Later still, Lagerstrom proposed his model problem to illustrate the mathematical ideas and techniques used by Kaplun in the asymptotic treatment of low Reynolds number flow, see [Kap57, Lag66, KL57]. In its simplest formulation, the model is given by the nonlinear, non-autonomous second-order boundary value problem

$$\ddot{u} + \frac{n-1}{x}\dot{u} + u\dot{u} = 0 \tag{1}$$

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with boundary conditions

$$u(\varepsilon) = 0, \qquad u(\infty) = 1. \tag{2}$$

Here $n \in \mathbb{N}$, $0 < \varepsilon \leq x \leq \infty$, and the overdot denotes differentiation with respect to x. Heuristically speaking, (1),(2) models slow incompressible viscous flow in n dimensions, scaled in such a way that the dependence of u on ε (the analogue of the Reynolds number) occurs through the inner boundary condition. Here, $u(\varepsilon) = 0$ corresponds to the no-slip condition at the surface of an "n-sphere" of diameter ε, whereas $u(\infty) = 1$ requires the "flow" to be uniform far away from the "n-sphere". We focus on the physically relevant cases $n = 2$ and $n = 3$ corresponding to flow around a cylinder and a sphere, respectively. This problem, which is not a model in the physical sense, but a mathematical model equation, was analyzed by Lagerstrom himself and many others, see e.g. [Bus71, CFL78, HTB90, KW96, KWK95, Lag88, RS75, Ski81], and the references therein. While it displays similar qualitative properties as the original fluid dynamical problem, Lagerstrom's model is analytically much simpler, owing largely to the fact that it is an ordinary rather than a partial differential equation. Both the original problem of viscous flow past a solid at low Reynolds number and Lagerstrom's model example have been quite influential for the development of singular perturbation theory in general and of the method of matched asymptotic expansions in particular.
More recently, an alternative approach to singularly perturbed problems known as *geometric singular perturbation theory* has been developed. This approach is based on methods from the theory of dynamical systems, in particular on invariant manifold theory. In this context, outer solutions and their expansions find a geometric explanation in terms of slow center-like manifolds which depend smoothly on the singular perturbation parameter. Standard exponential layer (inner) solutions are explained geometrically as invariant foliations of stable or unstable manifolds of slow center-like manifolds, which again depend smoothly on the singular perturbation parameter, see [Fen79] or [Jon95].
However, this well-developed geometric theory does not apply at points where normal hyperbolicity is lost, i.e., at points where the slow manifold ceases to be exponentially attractive respectively repelling.
Recently, it has been possible to extend the geometric approach to the case when normal hyperbolicity fails due to a single zero eigenvalue, a situation which arises frequently in applications, e.g. in relaxation oscillations. This advance has been possible due to the use of the blow-up method [DR91, Dum93, DR96]. Blow-up can be described as a sophisticated rescaling which allows one to identify the dominant scales in various regions near a singularity. In particular, the blow-up method has been used for a detailed analysis of the simple fold problem, see [KS01, vGKS]. In these works, slow manifolds are continued beyond the fold point. Additionally, the complicated structure of the corresponding asymptotic expansions is explained and an algorithm to compute them is given.
The aim of the present work and its sequel [PS] is to analyze Lagerstrom's model problem in a similar spirit.

The paper is organized as follows. In Section 2, a brief introduction to Lagerstrom's model equation and its analysis by means of matched asymptotic expansions is given. Section 3 contains a dynamical systems reformulation of the Lagerstrom model. The governing equations are rewritten as an autonomous system of ordinary differential equations. A solution of the original boundary value problem is seen to correspond to an orbit connecting an (unknown) point in a one-dimensional manifold $\mathcal{V}$ representing the boundary condition at $x = \varepsilon$ to a degenerate equilibrium Q which corresponds to the boundary condition at $x = \infty$. In the limit $\varepsilon = 0$, Q becomes even more degenerate, which – at least partially – explains the singular perturbation nature of the problem. A singular orbit Γ is identified which connects the manifold $\mathcal{V}$ to the equilibrium Q for $\varepsilon = 0$. To resolve this singular behaviour a blow-up transformation is introduced. In Section 4, the dynamics of the blown-up problem is analyzed in detail. In the blown-up system, existence and uniqueness of solutions for Lagerstrom's model is proved by carefully tracking the manifold $\mathcal{V}$ of boundary values along the singular orbit Γ to show that it intersects transversely the stable manifold of Q. In most parts of the analysis, one has to distinguish between the cases $n = 3$ and $n = 2$, the latter being difficult due to its more degenerate nature.

2. Lagerstrom's model equation

By introducing the inner (stretched) variable

$$\xi = \frac{x}{\varepsilon} \tag{3}$$

in (1),(2), we obtain the equivalent formulation

$$u'' + \frac{n-1}{\xi}u' + \varepsilon u u' = 0 \tag{4}$$

of Lagerstrom's model equation, with $1 \le \xi \le \infty$ and boundary conditions

$$u(1) = 0, \qquad u(\infty) = 1; \tag{5}$$

here the prime denotes differentiation with respect to ξ.

Lagerstrom's model equation is a classical example of a singularly perturbed problem, as the solution obtained by setting $\varepsilon = 0$ in (4) is not a uniformly valid approximation to the solution of (4) for $1 \le \xi \le \infty$. Moreover, it shows that having the small parameter ε multiply the highest derivative in a differential equation is not a necessary condition for a problem to be singular. In what is to come, we will restrict ourselves to the cases $n = 2$ and $n = 3$, which correspond to the physically relevant settings of flow in two and three dimensions, respectively.

We briefly discuss Lagerstrom's analysis of (1),(2) based on the method of matched asymptotic expansions. Let us first turn to $n = 3$: by assuming a (regular) perturbation expansion for u of the form

$$u(\xi, \varepsilon) = u_0(\xi) + \varepsilon u_1(\xi) + \dots, \tag{6}$$

one obtains with ξ fixed as $\varepsilon \to 0$

$$u_0'' + \frac{2}{\xi}u_0' = 0, \tag{7a}$$

$$u_1'' + \frac{2}{\xi}u_1' = -u_0 u_0'; \tag{7b}$$

in analogy with the fluid dynamical problem, (7a) is called the *Stokes equation.*

With $u_0 = 0$ for $\xi = 1$ and $u_0 \to 1$ for $\xi \to \infty$, the leading term of the inner approximation (6) is

$$u_0 = 1 - \frac{1}{\xi}; \tag{8}$$

the solution of (7b) which satisfies $u_1 = 0$ at $\xi = 1$ is given by

$$u_1 = -\left(1 + \frac{1}{\xi}\right)\ln\xi + \alpha\left(1 - \frac{1}{\xi}\right); \tag{9}$$

however, no choice of the constant α can prevent u_1 from being logarithmically infinite for $\xi \to \infty$. This is the analogue of the fluid dynamical *Whitehead paradox*. Thus, the naive expansion (6) is not uniformly valid for ξ large; this forces one to apply the rescaling in (3).

For $\xi = \mathcal{O}(\varepsilon^{-1})$, (6) and (9) imply $u = 1 + \mathcal{O}(\varepsilon\ln\varepsilon)$, which suggests to replace (6) by an expansion of the form

$$u(\xi,\varepsilon) = 1 - \frac{1}{\xi} + \varepsilon\ln\varepsilon\,\tilde{u}_1(\xi) + \varepsilon u_1(\xi) + \ldots \tag{10}$$

for ξ fixed as $\varepsilon \to 0$.

To approximate solutions of (1) for $x = \varepsilon\xi$ fixed as $\varepsilon \to 0$, one uses the outer expansion

$$u(x,\varepsilon) = U_0(x) + \varepsilon\ln\varepsilon\,\widetilde{U}_1(x) + \varepsilon U_1(x) + \ldots \tag{11}$$

which is akin to the Oseen expansion in the fluid dynamical problem. The leading order term satisfying (1) is found to be $U_0 = 1$. $\widetilde{U}_1$ has to satisfy the homogeneous Oseen equation

$$\frac{d^2\widetilde{U}_1}{dx^2} + \left(\frac{2}{x} + 1\right)\frac{d\widetilde{U}_1}{dx} = 0 \tag{12}$$

with $\widetilde{U}_1(\infty) = 0$; the same is true of U_1. Equation (12) is linear; its solution can be given in terms of certain exponential integrals, with the constants left to be determined by matching.

For $n = 2$, the situation is even more involved: the same intuitive reasoning as before yields

$$u_0 = \alpha\ln\xi \tag{13}$$

for the leading term of the inner approximation. Obviously, the condition at infinity cannot be satisfied with any choice of α (*Stokes paradox*). Nevertheless, the rescaling in (3) is applicable again, which implies that the troublesome condition at infinity is in the region of x fixed as $\varepsilon \to 0$. As u must be $\mathcal{O}(1)$ there, (6) and (13) imply that $\alpha = \mathcal{O}\left((\ln\varepsilon)^{-1}\right)$, which suggests asymptotic expansions

$$u(\xi,\varepsilon) = -\frac{1}{\ln\varepsilon}u_0(\xi) + \frac{1}{(\ln\varepsilon)^2}u_1(\xi) + \ldots \tag{14}$$

and

$$u(x,\varepsilon) = 1 - \frac{1}{\ln\varepsilon}U_1(x) + \frac{1}{(\ln\varepsilon)^2}U_2(x) + \ldots, \tag{15}$$

respectively. As for $n = 3$, matching these expansions is still possible, although the overlap domain is now very small, see [LC72]. This is in essence Kaplun's resolution of the Stokes paradox: the Stokes solution is an inner solution which must satisfy a matching condition, but not necessarily the boundary condition at infinity.

Rigorous results for Lagerstrom's model equation have been obtained by several workers by a variety of methods. Existence and uniqueness of solutions was shown in [RS75] by transforming (4) into a pair of integral equations and applying a contraction mapping theorem. Hsiao [Hsi73] gave a rigorous discussion of existence for $n = 2$ and $\varepsilon \to 0$, whereas Cole [Col68] utilized an invariance group of (1) to obtain a similar result. In [CFL78] a related initial value problem was transformed into an integral equation, which was then shown to have a unique solution by constructing suitable super- and subsolutions. Hunter et al. [HTB90] proved that the so-called Oseen iteration, an iterative scheme based on using the outer approximation throughout, converges for all ε to a unique solution.

***Remark* 1** (Logarithmic switchback)**.** The introduction of an $\varepsilon \ln \varepsilon$-term in (6) is unexpected, as it is not directly forced by the equation, but by the matching. Perturbation problems in which the small parameter ε (but not $\ln \varepsilon$) occurs in the formulation of the problem, whereas $\ln \varepsilon$ occurs in the asymptotic expansion, have been encountered conspicuously often in the resolution of paradoxes in problems of fluid dynamics. The phenomenon is known as *logarithmic switchback*, see [Lag88] for further details. □

***Remark* 2.** A generalization of (1) to arbitrary integral (and even real) dimensions is feasible and has indeed been undertaken by several workers, see e.g. [LR84]. Our approach applies for any $n \in \mathbb{R}$, $n \geq 2$, as well, with only a few minor changes required.

Notably, the form of the simpler inner expansion (6) depends even more critically on the value of n than the outer expansion. The larger n is, the further the occurrence of switchback terms is postponed; Stokes' paradox is thus only delayed, as it will always occur sooner or later. In particular, there is no switchback for n irrational. □

3. A dynamical systems approach

3.1. Our strategy. We will employ a shooting argument to prove existence and (local) uniqueness of solutions to the boundary value problem (1),(2). To that end, we rewrite Lagerstrom's model problem as an equivalent autonomous first-order dynamical system. As is usual in geometric singular perturbation theory, the starting point of the analysis are the equations on the fast (inner) scale, i.e., (4). We replace $\xi \in [1, \infty)$ by $\eta := \xi^{-1} \in (0, 1]$; $\xi' = 1$ implies the equation $\eta' = -\eta^2$. By setting $u' = v$, we obtain the system

$$
\begin{aligned}
u' &= v, \\
v' &= -(n-1)\eta v - \varepsilon u v, \\
\eta' &= -\eta^2
\end{aligned}
\tag{16}
$$

with boundary conditions

$$
u(1) = 0, \qquad \eta(1) = 1, \qquad u(\infty) = 1;
\tag{17}
$$

note that (17) in fact entails $\eta(\infty) = 0$ and $v(\infty) = 0$ for the solution to (16), whereas $v(1)$ remains yet to be determined.

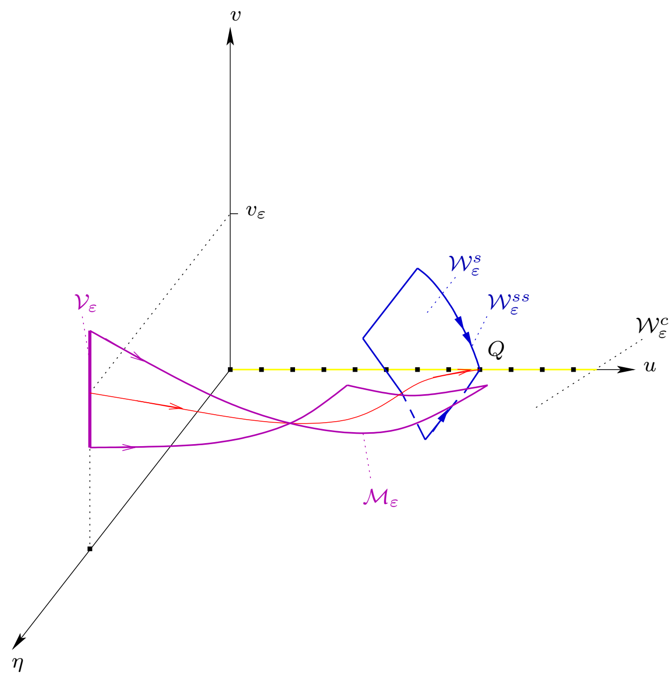

FIGURE 1. Geometry of system (16) for $\varepsilon > 0$ fixed.

We define the manifold $\mathcal{V}_\varepsilon$ by[1]

$$\mathcal{V}_\varepsilon := \left\{(0, v, 1) \,\middle|\, v \in [\underline{v}, \overline{v}]\right\} \tag{18}$$

with $0 \le \underline{v} < \overline{v} < \infty$ and the point Q by $Q := (1, 0, 0)$. Note that $\mathcal{V}_\varepsilon$ is a manifold of possible inner boundary values for (16). Moreover, one finds that Q is in fact an equilibrium of (16); indeed, one obtains a whole line of equilibria ℓ given by $\ell := \left\{(u, 0, 0) \,\middle|\, u \in \mathbb{R}\right\}$. The linearization of (16) at any such point is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\varepsilon u & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

which implies

Lemma 3.1. *For $\varepsilon > 0$, the eigenvalues of (19) are 0 and $-\varepsilon u$, where the multiplicity of 0 is two. The corresponding eigenspaces are*

$$\operatorname{span}\left\{(1, 0, 0)^T, (0, 0, 1)^T\right\}, \quad \operatorname{span}\left\{(1, -\varepsilon u, 0)^T\right\}. \tag{20}$$

For $\varepsilon = 0$, the multiplicity of 0 is three, with the eigenspace being

$$\operatorname{span}\left\{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\right\}; \tag{21}$$

here $(0, 1, 0)^T$ is a generalized eigenvector.

Standard results from invariant manifold theory yield

[1] Note that the subscript ε is superfluous here, but is needed to ensure consistency of notation later on.

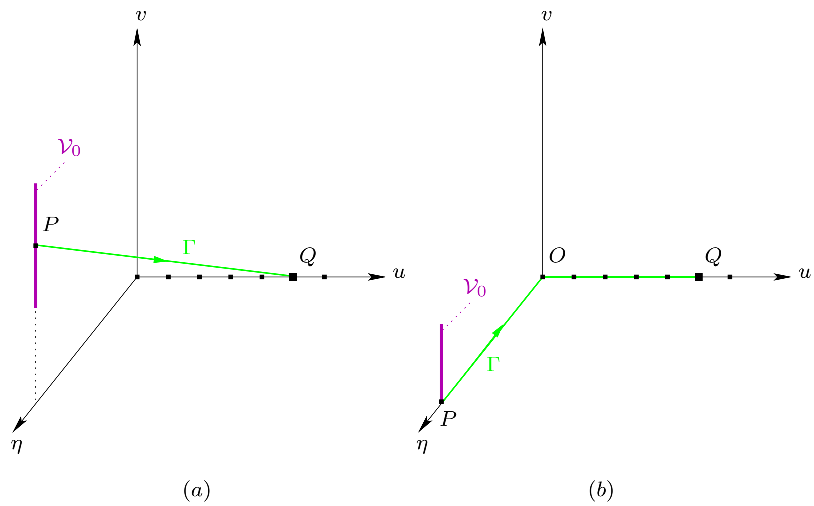

FIGURE 2. Geometry of (16) for $\varepsilon = 0$ and (a) $n = 3$, respectively, (b) $n = 2$.

Proposition 3.2. *Let $k \in \mathbb{N}$ be arbitrary, and let $\varepsilon > 0$.*

(1) *There exists an attracting two-dimensional center manifold $\mathcal{W}^c_\varepsilon$ of (16) which is given by $\{v = 0\}$.*
(2) *For $|u-1|$, v, and η sufficiently small, there is a stable invariant $\mathcal{C}^k$-smooth foliation $\mathcal{F}^s_\varepsilon$ with base $\mathcal{W}^c_\varepsilon$ and one-dimensional $\mathcal{C}^k$-smooth fibers.*

Proof. The first assertion is obvious from Lemma 3.1 and the fact that $\{v = 0\}$ is an invariant subspace for (16); the second assertion follows from standard invariant manifold theory (see e.g. [Fen79] or [CLW94]). □

Given $\varepsilon > 0$ fixed, one can thus define the stable manifold $\mathcal{W}^s_\varepsilon$ of Q as

$$\mathcal{W}^s_\varepsilon := \bigcup_{P \in \Upsilon} F^s_\varepsilon(P), \tag{22}$$

where $\Upsilon := \big\{(1, 0, \eta) \,\big|\, 0 \leq \eta \ll 1\big\}$, i.e., as a union of fibers $F^s_\varepsilon \in \mathcal{F}^s_\varepsilon$ with base points in the weakly stable orbit Υ, see Figure 1. Of particular importance is the fiber $F^s_\varepsilon(Q)$ with base point Q; note that by Lemma 3.1, $F^s_\varepsilon(Q)$ is tangent to $(1, -\varepsilon, 0)^T$ at Q.

***Remark* 3.** In fact, due to the simple structure of (16) for $\eta = 0$, $F^s_\varepsilon(Q)$ can be computed explicitly by writing e.g.

$$\frac{dv}{du} = -\varepsilon u \tag{23}$$

and solving for v to obtain

$$v(u) = \frac{\varepsilon}{2}\left(1 - u^2\right); \tag{24}$$

here we have used $v(1) = 0$. □

We will in the following write $\mathcal{W}^{ss}_\varepsilon$ instead of $F^s_\varepsilon(Q)$ to stress that $F^s_\varepsilon(Q)$ is the one-dimensional strongly stable manifold of Q.

A solution of the boundary value problem (16),(17) corresponds to a forward orbit starting in $\mathcal{V}_\varepsilon$ and converging to Q as $\xi \to \infty$. Hence, existence and uniqueness of solutions will follow by showing that the saturation of $\mathcal{V}_\varepsilon$ under the flow defined by (16), which we call $\mathcal{M}_\varepsilon := \mathcal{V}_\varepsilon \cdot [1,\infty)$, intersects $\mathcal{W}^s_\varepsilon$ in a unique orbit; here, the dot denotes the application of the flow induced by (16).

For $\varepsilon = 0$, it is straightforward to obtain *singular orbits* connecting $\mathcal{V}_0$ to Q. It is these orbits we will use as templates for orbits of the full problem ($\varepsilon > 0$). The case $n = 3$ is the simpler one, as the forward orbit

$$\gamma := \big\{ \big(1-\eta, \eta^2, \eta\big) \,\big|\, \eta \in (0,1] \big\} \tag{25}$$

through $P := (0,1,1)$ obtained by solving (16) for $\varepsilon = 0$ is asymptotic to Q. We thus define the singular orbit Γ by

$$\Gamma := \gamma \cup \{Q\}, \tag{26}$$

see Figure 2. For $n = 2$, the situation is more involved: remember that for $n = 2$, there is no solution to (16),(17) for $\varepsilon = 0$. However, a singular orbit Γ can still be defined: let $P := (0,0,1)$, and let γ denote the orbit

$$\gamma := \big\{ (0,0,\eta) \,\big|\, \eta \in (0,1] \big\} \tag{27}$$

through P, which is forward asymptotic to the origin O. Then,

$$\Gamma := \gamma \cup \{O\} \cup \big\{ (u,0,0) \,\big|\, u \in (0,1) \big\} \cup \{Q\}. \tag{28}$$

For $n = 2$, Γ thus contains a segment of the line of equilibria ℓ, which accounts for the complicated nature of the problem.

We now proceed as follows to prove existence and uniqueness of solutions to (16): we track $\mathcal{M}_\varepsilon$ through phase space and show that it intersects transversely the stable manifold $\mathcal{W}^s_\varepsilon$ of Q (see again Figure 1). Due to the fact that we are only interested in ε small, we are going to take a perturbational approach, i.e., we intend to set $\varepsilon = 0$ in (16) and track $\mathcal{M}_0$ along Γ under the resulting flow. For $\varepsilon = 0$, however, the equations in (16) are even more degenerate than they are for $\varepsilon > 0$, see Lemma 3.1. Due to the non-hyperbolic character of the problem for $\varepsilon = 0$, there is no stable foliation $\mathcal{F}^s_0$; hence, a stable manifold $\mathcal{W}^s_0$ does not exist, either. We therefore have to modify our approach. To that end, we extend (16) by appending the (trivial) equation $\varepsilon' = 0$, obtaining

$$\begin{aligned} u' &= v, \\ v' &= -(n-1)\eta v - \varepsilon u v, \\ \eta' &= -\eta^2, \\ \varepsilon' &= 0 \end{aligned} \tag{29}$$

in extended phase space, where the boundary conditions are still given by (17). Contrary to the above, the parameter ε is not fixed now, but is allowed to vary in an interval $[0,\varepsilon_0]$ with $\varepsilon_0 > 0$ small. Correspondingly, for the extended system (29) we define the manifolds $\mathcal{V}$ and $\mathcal{M}$ by $\mathcal{V} := \bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathcal{V}_\varepsilon \times \{\varepsilon\}$ and $\mathcal{M} := \bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathcal{M}_\varepsilon \times \{\varepsilon\}$, respectively. We will see that by using blow-up, we will be able to define stable manifolds $\mathcal{W}^{ss}$ and $\mathcal{W}^s$ in a smooth way down to $\varepsilon = 0$.

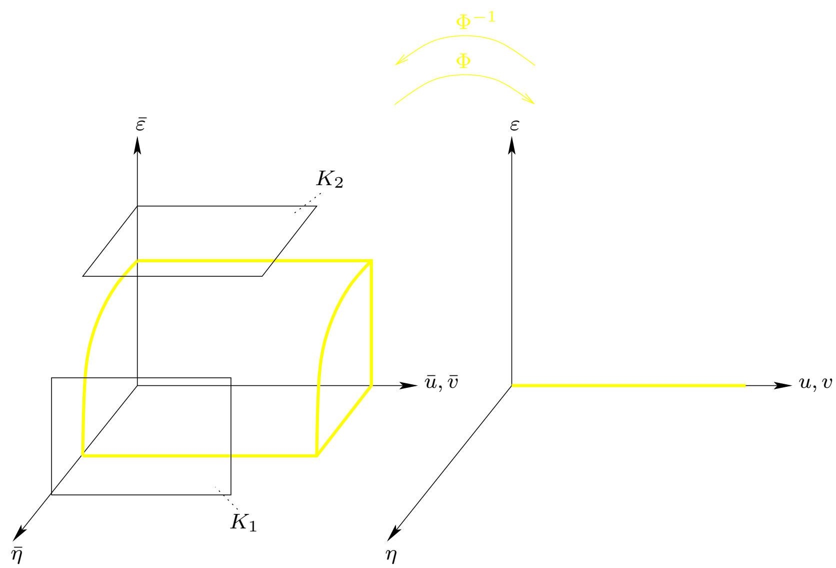

FIGURE 3. The blow-up transformation Φ.

***Remark* 4.** Though Lagerstrom's model equation is a singular perturbation problem, it is not strictly so in the sense of [Fen79]. Indeed, the dynamics of (29) is to be characterized as center-like rather than slow-fast. □

3.2. The blow-up transformation. To analyze the dynamics of (29) near the line $\ell := \big\{(u, 0, 0, 0) \,\big|\, u \in \mathbb{R}^+\big\}$ of equilibria[2] of (29), we introduce a *(polar) blow-up transformation*

$$\Phi : \begin{cases} \mathbb{R} \times B \to \mathbb{R}^4, \\ (\bar{u}, \bar{v}, \bar{\eta}, \bar{\varepsilon}, \bar{r}) \mapsto (\bar{u}, \bar{r}\bar{v}, \bar{r}\bar{\eta}, \bar{r}\bar{\varepsilon}) \end{cases} \tag{30}$$

with $B := \mathbb{S}^2 \times \mathbb{R}$. Here, $\mathbb{S}^2$ denotes the two-sphere in $\mathbb{R}^3$, i.e., $\mathbb{S}^2 = \big\{\big(\bar{v}, \bar{\eta}, \bar{\varepsilon}\big) \,\big|\, \bar{v}^2 + \bar{\eta}^2 + \bar{\varepsilon}^2 = 1\big\}$. Note that obviously $\Phi^{-1}(\ell) = \mathbb{R} \times \mathbb{S}^2 \times \{0\}$, which is the blown-up locus obtained by setting $\bar{r} = 0$. Moreover, for $\bar{r} \neq 0$, i.e., away from $\Phi^{-1}(\ell)$, Φ is a $\mathcal{C}^\infty$-diffeomorphism. We will only be interested in $\bar{r} \in [0, r_0]$ with $r_0 > 0$ small. The reason for introducing (30) is that degenerate equilibria, such as those in ℓ, can in many cases be neatly analyzed by means of blow-up techniques, see [Dum93]. The blow-up is a (singular) coordinate transformation whereby the degenerate equilibrium is blown up to some n-sphere. Transverse to the sphere and even on the sphere itself one often gains enough hyperbolicity to allow a complete analysis by standard techniques. For planar vector fields the method is widely known, see e.g. [GH83]; not unexpectedly, however, difficulties mount with rising dimension. A general discussion of blow-up can be found in [DR91]. The analysis of non-hyperbolic points in singular perturbation problems was initiated by Dumortier and Roussarie, see [Dum93, DR96], and was further developed in [KS01, vGKS]. We refer to these works for an introduction and more background material.

[2]We will in the following restrict ourselves to $u \geq 0$, due to the boundary conditions imposed.

The vector field on $\mathbb{R} \times B$, which is induced by the vector field corresponding to (29), is best studied by introducing different charts for the manifold $\mathbb{R} \times B$. In what is to come, it suffices to consider two charts K_1 and K_2 corresponding to $\bar{\eta} > 0$ and $\bar{\varepsilon} > 0$ in (30), respectively, see Figure 3. The coordinates $(u_1, v_1, r_1, \varepsilon_1)$ in K_1 are given by

$$u_1 = \bar{u}, \quad v_1 = \bar{v}\bar{\eta}^{-1}, \quad r_1 = \bar{r}\bar{\eta}, \quad \varepsilon_1 = \bar{\varepsilon}\bar{\eta}^{-1} \tag{31}$$

for $\bar{\eta} > 0$. Similarly, for (u_2, v_2, η_2, r_2) in K_2 one obtains

$$u_2 = \bar{u}, \quad v_2 = \bar{v}\bar{\varepsilon}^{-1}, \quad \eta_2 = \bar{\eta}\bar{\varepsilon}^{-1}, \quad r_2 = \bar{r}\bar{\varepsilon}, \tag{32}$$

where $\bar{\varepsilon} > 0$. We will see that these two charts correspond precisely to the inner and outer regions in the method of matched asymptotic expansions.

***Remark* 5** (Notation)**.** Let us introduce the following notation: for any object $\square$ in the original setting, let $\overline{\square}$ denote the corresponding object in the blow-up; in charts K_i, $i = 1, 2$, the same object will appear as $\square_i$ when necessary. $\square$

In K_1, the blow-up transformation (30) is

$$\Phi_1 : \begin{cases} \mathbb{R}^4 \to \mathbb{R}^4, \\ (u_1, v_1, r_1, \varepsilon_1) \mapsto (u_1, r_1 v_1, r_1, r_1\varepsilon_1), \end{cases} \tag{33}$$

which is a *directional blow-up* in the direction of positive η. With

$$u = u_1, \quad v = r_1 v_1, \quad \eta = r_1, \quad \varepsilon = r_1\varepsilon_1, \tag{34}$$

the blown-up vector field in K_1 is then given by

$$\begin{aligned} u_1' &= r_1 v_1, \\ v_1' &= (2-n) r_1 v_1 - r_1 \varepsilon_1 u_1 v_1, \\ r_1' &= -r_1^2, \\ \varepsilon_1' &= r_1 \varepsilon_1, \end{aligned} \tag{35}$$

which can be desingularized by setting $\frac{d}{d\xi} = r_1 \frac{d}{d\xi_1}$ in (35) and dividing out the common factor r_1 on both sides of the equations:

$$\begin{aligned} u_1' &= v_1, \\ v_1' &= (2-n) v_1 - \varepsilon_1 u_1 v_1, \\ r_1' &= -r_1, \\ \varepsilon_1' &= \varepsilon_1. \end{aligned} \tag{36}$$

This desingularization is necessary to obtain a non-trivial flow for $r_1 = 0$; it corresponds to a rescaling of time, leaving the phase portrait unchanged.
The equilibria of (36) are easily seen to lie in $\ell_1 := \{(u_1, 0, 0, 0) \,|\, u_1 \in \mathbb{R}^+\}$; the linearization there is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2-n & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

Depending on n, two cases have to be considered:

Lemma 3.3. *For $n = 3$, -1 is an eigenvalue of multiplicity two, whereas 0 and 1 are simple eigenvalues of* (37). *The corresponding eigenspaces are*

$$\operatorname{span}\left\{(0,0,1,0)^T,(1,-1,0,0)^T\right\}, \quad \operatorname{span}\left\{(1,0,0,0)^T\right\}, \quad \operatorname{span}\left\{(0,0,0,1)^T\right\}. \tag{38}$$

For $n = 2$, the multiplicity of 0 is two, with -1 and 1 simple and the eigenspaces given by

$$\operatorname{span}\left\{(1,0,0,0)^T,(0,1,0,0)^T\right\}, \quad \operatorname{span}\left\{(0,0,1,0)^T\right\}, \quad \operatorname{span}\left\{(0,0,0,1)^T\right\}; \tag{39}$$

here $(0,1,0,0)^T$ is a generalized eigenvector of the eigenvalue 0.

Proof. Computation. □

In chart K_2, the blow-up transformation (30) is given by

$$\Phi_2 : \begin{cases} \mathbb{R}^4 \to \mathbb{R}^4, \\ (u_2, v_2, \eta_2, r_2) \mapsto (u_2, r_2 v_2, r_2 \eta_2, r_2); \end{cases} \tag{40}$$

it follows that

$$u = u_2, \quad v = r_2 v_2, \quad \eta = r_2 \eta_2, \quad \varepsilon = r_2, \tag{41}$$

which is simply an ε-dependent rescaling of the original variables, since $r_2 = \varepsilon$. Given (41), we obtain for the blown-up vector field in K_2

$$\begin{aligned} u_2' &= r_2 v_2, \\ v_2' &= (1-n) r_2 \eta_2 v_2 - r_2 u_2 v_2, \\ \eta_2' &= -r_2 \eta_2^2, \\ r_2' &= 0. \end{aligned} \tag{42}$$

Desingularizing (dividing by r_2) once again yields

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= (1-n) \eta_2 v_2 - u_2 v_2, \\ \eta_2' &= -\eta_2^2, \\ r_2' &= 0; \end{aligned} \tag{43}$$

these equations are simple insofar as r_2 occurs only as a parameter. The equilibria of (43) are given by $\ell_2 := \left\{(u_2, 0, 0, r_2) \,\middle|\, u_2 \in \mathbb{R}^+, r_2 \in [0, r_0]\right\}$, with corresponding linearizations

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

Lemma 3.4. *The eigenvalues of* (44) *are 0 and $-u_2$, where the multiplicity of 0 is three. The corresponding eigenspaces are*

$$\operatorname{span}\left\{(1,0,0,0)^T,(0,0,1,0)^T,(0,0,0,1)^T\right\}, \quad \operatorname{span}\left\{(1,-u_2,0,0)^T\right\}. \tag{45}$$

Proof. Computation. □

Note that the line ℓ_2 corresponds exactly to the original ℓ, i.e., the point Q we are ultimately interested in is retrieved in chart K_2 after the blow-up. For the change of coordinates between charts K_1 and K_2 on their overlap domain we have the following result:

Lemma 3.5. *Let κ_{12} denote the change of coordinates from K_1 to K_2, and let $\kappa_{21} = \kappa_{12}^{-1}$ be its inverse. Then, κ_{12} is given by*

$$u_2 = u_1, \quad v_2 = v_1\varepsilon_1^{-1}, \quad \eta_2 = \varepsilon_1^{-1}, \quad r_2 = r_1\varepsilon_1, \tag{46}$$

and κ_{21} is given by

$$u_1 = u_2, \quad v_1 = v_2\eta_2^{-1}, \quad r_1 = r_2\eta_2, \quad \varepsilon_1 = \eta_2^{-1}. \tag{47}$$

Proof. Computation. □

For the computations which now follow it is convenient to define *sections* Σ_1^{in}, Σ_1^{out}, and Σ_2^{in} in K_1 and K_2, respectively, where

$$\Sigma_1^{in} := \big\{(u_1, v_1, r_1, \varepsilon_1) \,\big|\, u_1 \geq 0, v_1 \geq 0, \varepsilon_1 \geq 0, r_1 = \rho\big\}, \tag{48a}$$

$$\Sigma_1^{out} := \big\{(u_1, v_1, r_1, \varepsilon_1) \,\big|\, u_1 \geq 0, v_1 \geq 0, r_1 \geq 0, \varepsilon_1 = \delta\big\}, \tag{48b}$$

$$\Sigma_2^{in} := \big\{(u_2, v_2, \eta_2, r_2) \,\big|\, u_2 \geq 0, v_2 \geq 0, r_2 \geq 0, \eta_2 = \delta^{-1}\big\}, \tag{48c}$$

with $0 < \rho, \delta \ll 1$ arbitrary, but fixed; see Figure 4. Note that $\kappa_{12}\big(\Sigma_1^{out}\big) = \Sigma_2^{in}$. The shooting argument outlined in Section **3.1** is now carried out in the blown-up system, or, to be more precise, in charts K_1 and K_2. The sole reason for introducing (30) and considering (36) and (43) instead of (29), however, is that we have gained enough hyperbolicity to extend the argument down to and including $\varepsilon = 0$, i.e., to define the stable manifold $\overline{\mathcal{W}}^s$ of $\overline{Q} \in \bar{\ell}$ even for $\bar{r} = 0$. This follows from chart K_2, as the linearization of (43) at Q_2 has a negative eigenvalue irrespective of the value of r_2, in contrast to the linearization of the original (29) at Q. We will thus be able to track $\overline{\mathcal{V}}$ along the singular orbit $\overline{\Gamma}$ and show that the resulting manifold $\overline{\mathcal{M}}$ intersects $\overline{\mathcal{W}}^s$ transversely. This intersection will give the sought-after family of solutions to the boundary value problem (16),(17) for $\varepsilon \in (0, \varepsilon_0]$. The situation is illustrated in Figure 5.

4. Existence and uniqueness of solutions

In order to prove existence and uniqueness of solutions to (16),(17), we have to distinguish between the cases $n = 2$ and $n = 3$, due to the particularly degenerate structure of the problem for $n = 2$. In a first step, we consider the dynamics of (29) in charts K_1 and K_2 separately, which we then combine to obtain the main result of this paper:

Theorem 4.1. *For $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small and $n = 2, 3$, there exists a locally unique solution to* (16),(17) *close to the singular orbit* Γ.

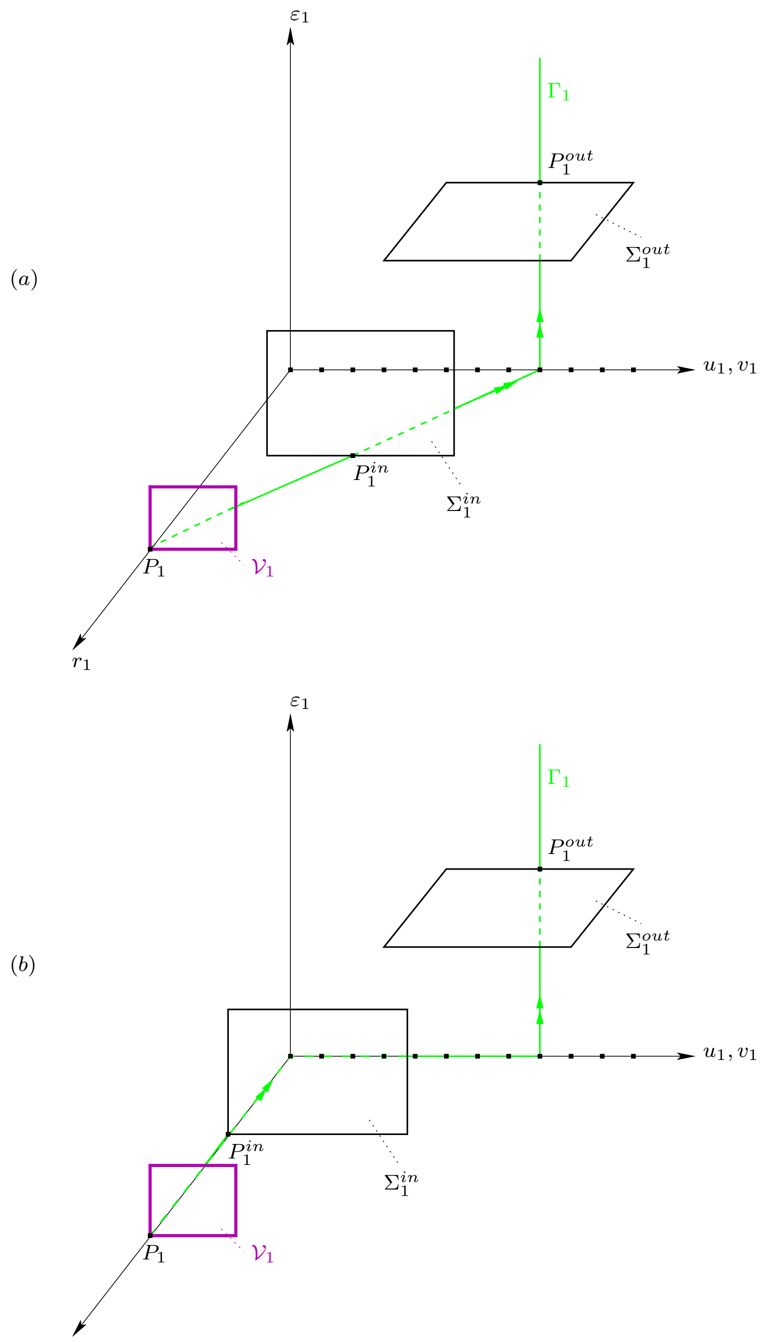

FIGURE 4. Geometry in chart K_1 for (a) $n = 3$ and (b) $n = 2$.

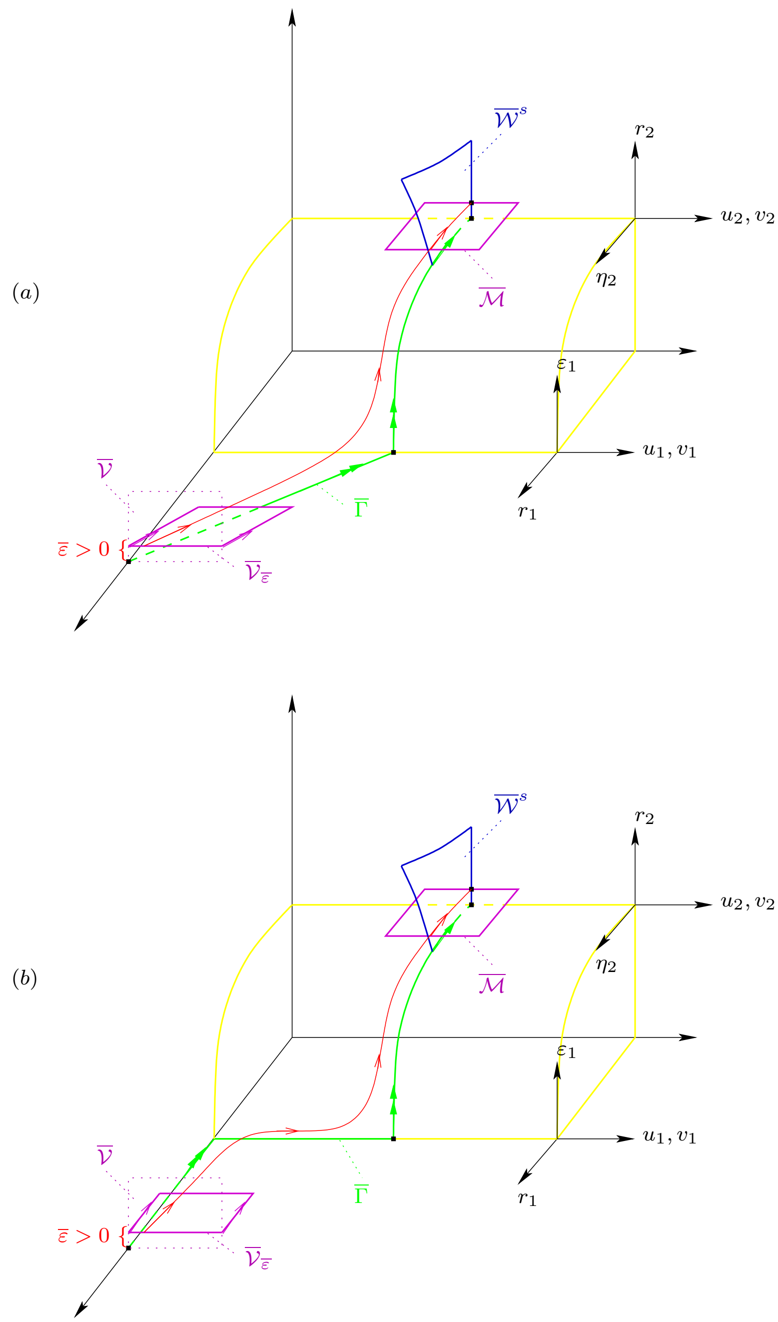

FIGURE 5. Geometry of the blown-up system for (a) $n = 3$ and (b) $n = 2$.

4.1. The case $n = 3$.

4.1.1. *Dynamics in chart K_1.* Let $\mathcal{V}_1$ denote the manifold $\overline{\mathcal{V}}$ in K_1, i.e.,

$$\mathcal{V}_1 := \left\{(0, v_1, 1, \varepsilon_1) \,\middle|\, |v_1 - 1| \le \alpha, \varepsilon_1 \in [0, \varepsilon_0]\right\} \tag{49}$$

for some $0 < \alpha < 1$. To obtain the singular orbit $\overline{\Gamma}$ in K_1, note that $\varepsilon = 0$ in (29) implies $\varepsilon_1 = 0$ in (36) due to $\varepsilon = r_1\varepsilon_1$ and $r_1 > 0$ in (49). In general, given the initial conditions[3]

$$(u_1, v_1, r_1, \varepsilon_1)^T(0) = (0, v_{1_0}, 1, 0)^T, \tag{50}$$

equations (36) can easily be solved explicitly:

$$(u_1, v_1, r_1, \varepsilon_1)^T(\xi_1) = \left(v_{1_0}\left(1 - e^{-\xi_1}\right), v_{1_0}e^{-\xi_1}, e^{-\xi_1}, 0\right)^T. \tag{51}$$

Let $\gamma_1(\xi_1)$ now denote the orbit corresponding to $P_1 = (0, 1, 1, 0)$, i.e., to $v_{1_0} = 1$ in (51),

$$\gamma_1(\xi_1) := \left\{\left(1 - e^{-\xi_1}, e^{-\xi_1}, e^{-\xi_1}, 0\right) \,\middle|\, \xi_1 \in [0, \infty)\right\}, \tag{52}$$

and let $P_1^{in} := \gamma_1 \cap \Sigma_1^{in}$; note that $\gamma_1(\xi_1) \to Q_1 = (1, 0, 0, 0)$ as $\xi_1 \to \infty$. In analogy to Section 3, we thus define Γ_1 by

$$\Gamma_1 := \gamma_1 \cup \{Q_1\} \cup \left\{(1, 0, 0, \varepsilon_1) \,\middle|\, \varepsilon_1 \in (0, \infty)\right\}, \tag{53}$$

see Figure 6. With the *variational equations* of (36) along γ_1 given by

$$\begin{aligned}
\delta u_1' &= \delta v_1, \\
\delta v_1' &= -\delta v_1 - e^{-\xi_1}\left(1 - e^{-\xi_1}\right)\delta\varepsilon_1, \\
\delta r_1' &= -\delta r_1, \\
\delta\varepsilon_1' &= \delta\varepsilon_1,
\end{aligned} \tag{54}$$

we obtain the following result:

Proposition 4.2. *Let $T_{P_1}\mathcal{V}_1$ denote the tangent space to $\mathcal{V}_1$ at P_1, let $\mathbf{t}_{P_1} \in T_{P_1}\mathcal{V}_1$ be the tangent direction spanned by*

$$(\delta u_1, \delta v_1, \delta r_1, \delta\varepsilon_1)^T(0) = (0, 1, 0, 0)^T, \tag{55}$$

and let $\mathbf{t}_{\gamma_1} \in T_{\gamma_1}\mathcal{M}_1$ be the solution of (54) *corresponding to* (52). *Then, $\mathbf{t}_{P_1^{in}} \in T_{P_1^{in}}\mathcal{M}_1$ is given by*

$$(\delta u_1, \delta v_1, \delta r_1, \delta\varepsilon_1)^T(-\ln\rho) = (1 - \rho, \rho, 0, 0)^T, \tag{56}$$

where ρ is as in the definition of Σ_1^{in}.

Proof. For the proof, note that clearly $\delta r_1 \equiv 0 \equiv \delta\varepsilon_1$. The equations in (54) then reduce to

$$\begin{aligned}
\delta u_1' &= \delta v_1, \\
\delta v_1' &= -\delta v_1,
\end{aligned} \tag{57}$$

which can be solved to give

$$(\delta u_1, \delta v_1)^T(\xi_1) = \left(1 - e^{-\xi_1}, e^{-\xi_1}\right)^T. \tag{58}$$

[3]It is no restriction to set $\xi_{1_0} = 0$ here; indeed, this can always be achieved by choosing the integration constant in $\xi_1(\xi) = \int r_1(\xi)d\xi$ appropriately.

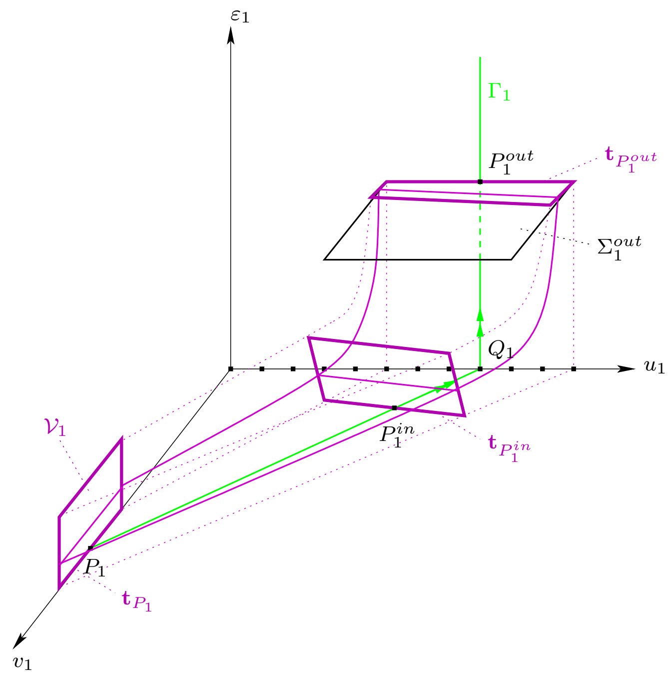

FIGURE 6. Dynamics in chart K_1 for $n = 3$.

From (52) it follows that

$$\rho = e^{-\xi_1} \tag{59}$$

in Σ_1^{in}, which completes the proof. □

***Remark* 6.** For reasons which will become clear later on, the evolution of the tangent direction to $\mathcal{V}_1$ spanned by $(\delta u_1, \delta v_1, \delta r_1, \delta\varepsilon_1)^T(0) = (0, 0, 0, 1)^T$ is of no relevance to us and thus is not considered here. □

The analysis of the transition of $\mathcal{M}_1$ from Σ_1^{in} to Σ_1^{out} past the line ℓ_1 of partially hyperbolic equilibria is more subtle. For hyperbolic equilibria, normal form transformations combined with cut-off techniques can be used to eliminate higher-order terms, see [Ste58]. For partially hyperbolic equilibria satisfying certain non-resonance conditions, a transformation to standard form can still be found, see [Tak71] or [Bon96]. By Lemma 3.3, however, the eigenvalues of (37) obviously are in *resonance* both for $n = 3$ and for $n = 2$. Hence, the above techniques do not apply. We therefore have to proceed directly, i.e., by estimation, to obtain bounds on u_1 and v_1 in Σ_1^{out}. In fact, it is these resonances which are responsible for the occurrence of logarithmic switchback terms in the Lagerstrom model. This will become more evident in the upcoming paper [PS], where asymptotic expansions for the solutions to (16),(17) as specified in Theorem 4.1 will be derived.

***Remark* 7.** Note that for n irrational in (29), the resonances are destroyed, which explains the absence of logarithmic switchback in Lagerstrom's model then. □

The following simple observation will prove quite useful:

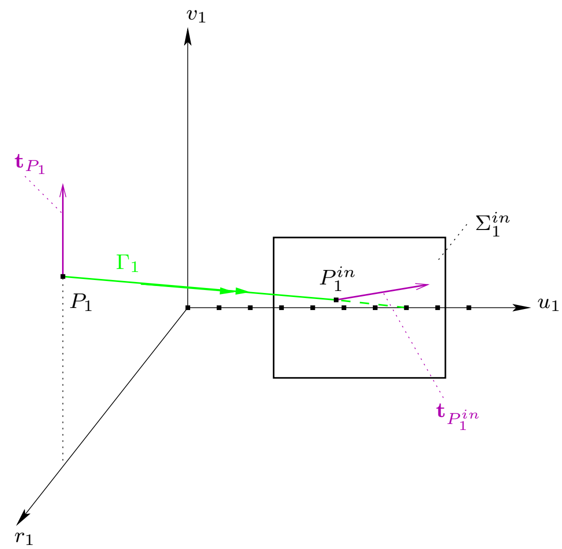

FIGURE 7. Evolution of $\mathbf{t}_{P_1}$ under the flow of (54).

Lemma 4.3. *For any $u_{1_0}, v_{1_0} \geq 0$, the solutions $u_1(\xi_1)$ and $v_1(\xi_1)$ to (36) can be estimated as follows:*

$$\begin{aligned} u_{1_0} &\leq u_1(\xi_1) \leq u_{1_0} + v_{1_0}\left(1 - e^{-\xi_1}\right), \\ 0 &\leq v_1(\xi_1) \leq v_{1_0} e^{-\xi_1}. \end{aligned} \tag{60}$$

Proof. For[4] $v_0 \geq 0$, $v(\xi) \geq 0$ follows from the invariance of $\{v = 0\}$ in (36). As

$$v' = -v - \varepsilon uv \leq -v, \tag{61}$$

integration yields $v(\xi) \leq v_0 e^{-\xi_1}$. Similarly, the estimate for $u(\xi)$ is obtained by integrating $0 \leq u' \leq v_0 e^{-\xi}$. □

Proposition 4.2 asserts that $\mathcal{M}_1$ is very much tilted in the direction of u_1 by the flow of (36): despite $\mathbf{t}_{P_1}$ being vertical, $\mathbf{t}_{P_1^{in}}$ is almost horizontal already, as δv_1 is almost annihilated during transport, whereas δu_1 is hugely expanded, see Figure 7. The next result shows that the transition from Σ_1^{in} to Σ_1^{out} only serves to make the tilt more pronounced, with δv_1 being even further contracted at the expense of δu_1:

Proposition 4.4. *Let $\Pi : \Sigma_1^{in} \to \Sigma_1^{out}$ denote the transition map for (36), and let $P_1^{out} := \Pi\big(P_1^{in}\big)$. Then, $D\Pi\big(\mathbf{t}_{P_1^{in}}\big) = \mathbf{t}_{P_1^{out}}$ is spanned by*

$$\left(\delta u_1^{out}, \delta v_1^{out}, \delta r_1^{out}, \delta \varepsilon_1^{out}\right)^T(\infty) = (1, 0, 0, 0)^T. \tag{62}$$

***Remark* 8.** Technically speaking, there is of course no transition at all past ℓ_1 for $\varepsilon_1 = 0$; hence Π and $D\Pi$ have to be defined by taking the limit $\varepsilon_1 \to 0$. The following proof will show that this limit is in fact well-defined. □

[4]Here and in most of the following proofs, we will omit the subscripts for the sake of readability.

Proof of Proposition 4.4. For $0 < \varepsilon^{in} < \frac{\varepsilon}{\rho}$, let $\tilde{P}^{in} := (\tilde{u}^{in}, \tilde{v}^{in}, \rho, \varepsilon^{in}) \in \mathcal{M} \cap \Sigma^{in}$, and let

$$\tilde{\Gamma}(\xi) := \left(\tilde{u}(\xi), \tilde{v}(\xi), \rho e^{-\xi}, \varepsilon^{in} e^{\xi}\right) \tag{63}$$

be the solution to (36) starting in $\tilde{P}^{in}$ (see Figure 8). The variational equations of (36) along $\tilde{\Gamma}$ are given by

$$\begin{aligned} \delta u' &= \delta v, \\ \delta v' &= -\varepsilon^{in} e^{\xi} \tilde{v} \delta u + \left(-1 - \varepsilon^{in} e^{\xi} \tilde{u}\right) \delta v - \tilde{u}\tilde{v}\delta\varepsilon, \\ \delta r' &= -\delta r, \\ \delta\varepsilon' &= \delta\varepsilon \end{aligned} \tag{64}$$

with initial conditions in $T_{\tilde{P}^{in}}\mathcal{M}$, i.e.,

$$(\delta u, \delta v, \delta r, \delta\varepsilon)^T(0) = (\delta u^{in}, \delta v^{in}, 0, 0)^T. \tag{65}$$

As before, $\delta r \equiv 0 \equiv \delta\varepsilon$, and we obtain

$$\begin{aligned} \delta u' &= \delta v, \\ \delta v' &= -\delta v - \varepsilon^{in} e^{\xi} (\tilde{v}\delta u + \tilde{u}\delta v). \end{aligned} \tag{66}$$

To prove our assertion, we proceed by plugging (66) into

$$\left(\frac{\delta u}{\delta v}\right)' = \frac{\delta u' \delta v - \delta u \delta v'}{(\delta v)^2}, \tag{67}$$

whence

$$z' = 1 + \left(1 + \varepsilon^{in} e^{\xi} \tilde{u}\right) z + \varepsilon^{in} e^{\xi} \tilde{v} z^2; \tag{68}$$

here we have set $z := \frac{\delta u}{\delta v}$. From Lemma 4.3 we conclude that $\tilde{u}, \tilde{v} \geq 0$. With $z^{in} = \left(\frac{\delta u}{\delta v}\right)^{in} > 0$ for ε^{in} sufficiently small, this gives $z' \geq 1$ as long as z remains bounded, i.e., as long as $\delta v > 0$; therefore

$$z(\xi) \geq \xi + z^{in}. \tag{69}$$

A similar argument for $w = z^{-1}$ shows that z indeed cannot become unbounded for finite ξ. As (63) yields

$$\delta = \varepsilon^{in} e^{\xi} \tag{70}$$

in Σ^{out}, the assertion now follows from

$$\left(\frac{\delta u}{\delta v}\right)^{out} \geq \ln \frac{\delta}{\varepsilon^{in}} \tag{71}$$

with $\varepsilon^{in} \to 0$. □

4.1.2. *Dynamics in chart K_2.* Let $Q_2 = (1, 0, 0, 0)$ in chart K_2. The following observation is crucial for everything that follows:

Lemma 4.5. *Let $k \in \mathbb{N}$ be arbitrary.*

1. *There exists an attracting three-dimensional center manifold $\mathcal{W}_2^c$ of* (43) *which is given by $\{v_2 = 0\}$.*
2. *For $|u_2 - 1|$, v_2, η_2, and r_2 sufficiently small, there is a stable invariant $\mathcal{C}^k$-smooth foliation $\mathcal{F}_2^s$ with base $\mathcal{W}_2^c$ and one-dimensional $\mathcal{C}^k$-smooth fibers.*

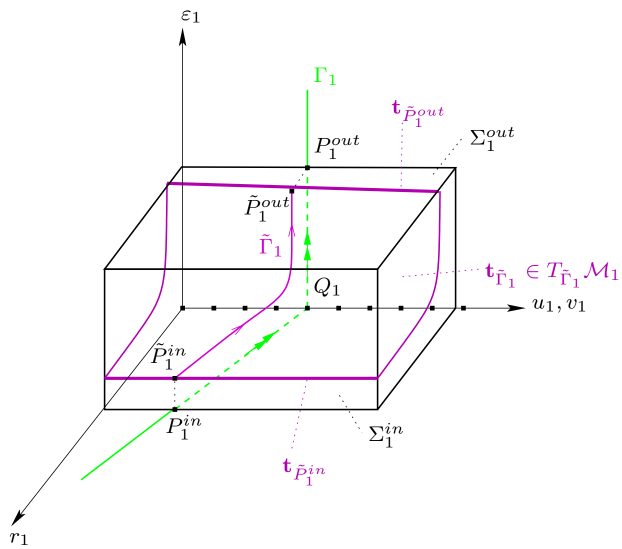

FIGURE 8. Illustration of the proof of Proposition 4.4.

Proof. The first assertion follows immediately from Lemma 3.4 and the fact that $\{v_2 = 0\}$ is obviously an invariant subspace for (43); the second assertion is obtained from invariant manifold theory (see e.g. [Fen79] or [CLW94]). $\square$

Let $F_2^s(Q_2) \in \mathcal{F}_2^s$ be the fiber emanating from Q_2; as in the original setting, we once again write $\mathcal{W}_2^{ss}$ for $F_2^s(Q_2)$. Indeed, note that for any $r_2 = \varepsilon \in [0, \varepsilon_0]$ fixed, Q_2 and $\mathcal{W}_2^{ss}$ correspond to the original Q and its stable fiber $\mathcal{W}_\varepsilon^{ss}$, respectively.

***Remark* 9.** Note that $\mathcal{W}_2^{ss}$ is known explicitly, as for $\eta_2 = 0$, (43) yields

$$\frac{dv_2}{du_2} = -u_2. \tag{72}$$

With $v_2(1) = 0$ we thus obtain

$$v_2(u_2) = \frac{1}{2}\left(1 - u_2^2\right); \tag{73}$$

hence $\mathcal{W}_2^{ss}$ is independent of n, as was to be expected. $\square$

Let the orbit γ_2 be defined by

$$\gamma_2(\xi_2) := \left\{ \left(1, 0, \xi_2^{-1}, 0\right) \,\middle|\, \xi_2 \in (0, \infty) \right\} \tag{74}$$

(note that $\gamma_2(\xi_2) \to Q_2$ as $\xi_2 \to \infty$), and let $\Gamma_2 := \gamma_2 \cup \{Q_2\}$; in fact, Γ_2 is precisely the continuation of Γ_1 in K_2. With Lemma 4.5 it then follows:

Proposition 4.6. *The manifold $\mathcal{W}_2^s$ defined by*

$$\mathcal{W}_2^s := \bigcup_{P_2 \in \Gamma_2} F_2^s(P_2) \tag{75}$$

is an invariant, $\mathcal{C}^k$-smooth manifold, namely the stable manifold of Q_2.

To obtain an approximation to $\mathcal{W}_2^s$ through its tangent bundle $T\mathcal{W}_2^s$ along γ_2, we consider the variational equations of (43). The latter are given by

$$\begin{aligned} \delta u_2' &= \delta v_2, \\ \delta v_2' &= -\left(\frac{2}{\xi_2}+1\right)\delta v_2, \\ \delta \eta_2' &= -\frac{2}{\xi_2}\delta\eta_2, \\ \delta r_2' &= 0, \end{aligned} \tag{76}$$

where $' = \frac{d}{d\xi_2}$ as before. Note that the first and second equation from (76) combined yield

$$\delta u_2'' = -\left(\frac{2}{\xi_2}+1\right)\delta u_2'; \tag{77}$$

this equation, which is precisely the (linear) *Oseen equation* from classical theory, has the solution

$$\delta u_2(\xi_2) = \alpha E_2(\xi_2) + \beta, \tag{78}$$

where[5]

$$E_k(z) := \int_z^\infty e^{-t}t^{-k}dt, \qquad z\in\mathbb{C},\ \Re(z)>0,\ k\in\mathbb{N}, \tag{79}$$

and α, β are some constants which are as yet undetermined. From Lemma 3.4 we know that the tangent direction $\mathbf{t}_{Q_2} \in T_{Q_2}\mathcal{W}_2^{ss}$ to $\mathcal{W}_2^{ss}$ is spanned by the vector

$$(\delta u_2, \delta v_2, \delta\eta_2, \delta r_2)^T(\infty) = (-1,1,0,0)^T. \tag{80}$$

The following proposition describes the evolution of $\mathbf{t}_{\Gamma_2} \in T_{\Gamma_2}\mathcal{W}_2^s$, which is $\mathbf{t}_{Q_2}$ extended along Γ_2 as $\xi_2 \to 0$ (see Figure 9):

Proposition 4.7. *Let $Q_2, \mathcal{W}_2^s$, and $\mathbf{t}_{\Gamma_2}$ be defined as above, and let $Q_2^{in} := \gamma_2 \cap \Sigma_2^{in}$. Then, $\mathbf{t}_{Q_2^{in}} \in T_{Q_2^{in}}\mathcal{W}_2^s$ is spanned by*

$$(\delta u_2, \delta v_2, \delta\eta_2, \delta r_2)^T(\delta) = \left(\delta u_2^{in}, 1, 0, 0\right)^T, \tag{81}$$

where

$$\delta u_2^{in} = \mathcal{O}(\delta). \tag{82}$$

Proof. As we know the solution to

$$\begin{aligned} \delta u' &= \delta v, \\ \delta v' &= -\left(\frac{2}{\xi}+1\right)\delta v \end{aligned} \tag{83}$$

to be

$$(\delta u, \delta v)^T(\xi) = \left(\alpha\int_\xi^\infty e^{-t}t^{-2}dt + \beta, -\alpha e^{-\xi}\xi^{-2}\right)^T, \tag{84}$$

we can now determine α and β from the condition that

$$\lim_{\xi\to\infty}\frac{\delta u}{\delta v}(\xi) = \lim_{\xi\to\infty}\left[-e^{\xi}\xi^2\int_\xi^\infty e^{-t}t^{-2}dt - \frac{\beta}{\alpha}e^{\xi}\xi^2\right] = -1. \tag{85}$$

[5]Here $\Re(z)$ denotes the real part of z.

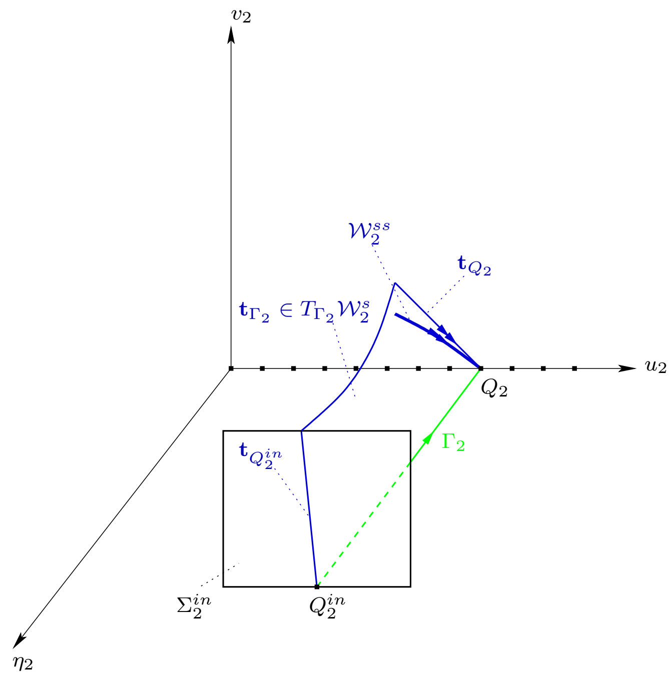

FIGURE 9. Geometry and dynamics in chart K_2.

An easy application of *de l'Hôspital's rule* to the first term in (85) shows that we have to require $\beta = 0$; to fix α, we demand that $\delta v = 1$ in section Σ^{in}. Reverting to our original subscripts and remembering that $\xi_2 = \eta_2^{-1}$, in Σ_2^{in} we therefore have $\xi_2 = \delta$, so that $P_2^{in} = \gamma_2(\delta)$ and

$$\alpha = -e^{\delta}\delta^2. \tag{86}$$

For δu_2, the substitution $\tau = \frac{t}{\delta}$ thus yields

$$\delta u_2 = -e^{\delta}\delta^2 \int_{\delta}^{\infty} e^{-t}t^{-2}dt = -e^{\delta}\delta \int_{1}^{\infty} e^{-\delta\tau}\tau^{-2}d\tau. \tag{87}$$

This latter integral, which we denote by

$$\tilde{E}_k(z) := \int_1^{\infty} e^{-z\tau}\tau^{-k}d\tau, \qquad z \in \mathbb{C},\ \Re(z) > 0,\ k \in \mathbb{N}, \tag{88}$$

and its properties are well known, see e.g. [AS64]:

Lemma 4.8. *For $|\arg z| < \pi$, $\tilde{E}_k(z)$ has the series expansion*

$$\tilde{E}_1(z) = -\gamma - \ln z - \sum_{i=1}^{\infty} \frac{(-z)^i}{i \cdot i!} \tag{89}$$

for $k = 1$ and

$$\tilde{E}_k(z) = \frac{(-z)^{k-1}}{(k-1)!}\left[-\ln z + \psi(k)\right] - \sum_{\substack{i=0 \\ i \neq k-1}}^{\infty} \frac{(-z)^i}{(i-k+1) \cdot i!} \tag{90}$$

for $k \geq 2$; here $\gamma = 0.5772\ldots$ is Euler's constant and

$$\psi(1) = -\gamma, \qquad \psi(k) = -\gamma + \sum_{i=1}^{k-1} \frac{1}{i}, \quad k \geq 2. \tag{91}$$

For $n = 3$, we thus obtain

$$\tilde{E}_2(z) = z\,[\ln z + \gamma - 1] - \sum_{\substack{i=0 \\ i\neq 1}}^{\infty} \frac{(-z)^i}{(i-1)\cdot i!}, \tag{92}$$

and the proof is completed by plugging (92) into (87) and collecting powers of δ. $\square$

4.2. The case $n = 2$.

4.2.1. *Dynamics in chart K_1.* As indicated already, our approach in this section will have to be quite different from the above, which is due to the complicated nature of the singular orbit Γ_1 for $n = 2$ as compared to $n = 3$. With

$$\gamma_1(\xi_1) := \big\{\big(0, 0, e^{-\xi_1}, 0\big) \,\big|\, \xi_1 \in [0, \infty)\big\} \tag{93}$$

denoting again the orbit through $P_1 = (0, 0, 1, 0)$ which is asymptotic to the origin, Γ_1 is given by

$$\Gamma_1 := \gamma_1 \cup \{O\} \cup \big\{(u_1, 0, 0, 0) \,\big|\, u_1 \in (0, 1)\big\} \cup \{Q_1\} \cup \big\{(1, 0, 0, \varepsilon_1) \,\big|\, \varepsilon_1 \in (0, \infty)\big\} \tag{94}$$

(see Figure 10). However, rather than investigating (36) for $\varepsilon_1 = 0$, as before, we now consider (36) with the perturbative terms $-\varepsilon_1 u_1 v_1$ omitted, which is

$$\begin{aligned} u_1' &= v_1, \\ v_1' &= 0, \\ r_1' &= -r_1, \\ \varepsilon_1' &= \varepsilon_1; \end{aligned} \tag{95}$$

here, the initial conditions are given by

$$(u_1, v_1, r_1, \varepsilon_1)^T(0) = (0, v_{1_0}, 1, \varepsilon_{1_0})^T. \tag{96}$$

The reason for considering (95) instead of (36) is that these equations can easily be solved, yielding

$$(u_1, v_1, r_1, \varepsilon_1)^T(\xi_1) = \big(v_{1_0}\xi_1, v_{1_0}, e^{-\xi_1}, \varepsilon_{1_0} e^{\xi_1}\big)^T; \tag{97}$$

in a second step, we will prove that (97) is in fact a good approximation to the corresponding solution to (36), which justifies our approach. Note that due to (34), we have $\varepsilon_{1_0} = \varepsilon$ in (97).[6] The manifold $\mathcal{V}_1$ of boundary conditions is given by

$$\mathcal{V}_1 := \big\{(0, v_1, 1, \varepsilon_1) \,\big|\, 0 \leq v_1 \leq \alpha, \varepsilon_1 \in [0, \varepsilon_0]\big\} \tag{98}$$

with $0 < \alpha < 1$. First, we show that for some suitable set $U_1^{out} \subset \Sigma_1^{out}$ containing $P_1^{out} := \Gamma_1 \cap \Sigma_1^{out}$ and any point $\tilde{P}_1^{out} \in U_1^{out}$, we can choose a $\tilde{P}_1 \in \mathcal{V}_1$ such that there is a solution of (95) passing through $\tilde{P}_1^{out}$:[7]

[6] In the following, we will use the two synonymously whenever there is no danger of confusion.

[7] Note that the size of U_1^{out} is restricted merely by the values of α and ε_0 in the definition of $\mathcal{V}_1$.

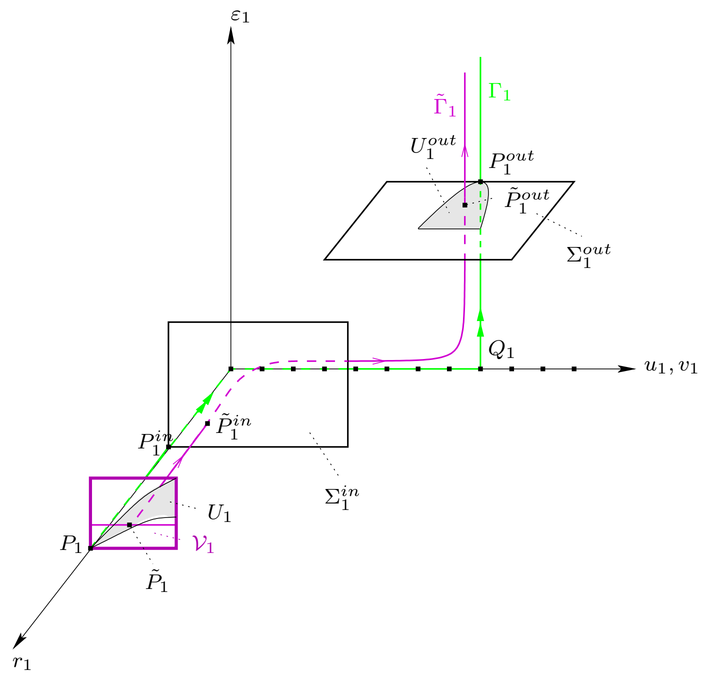

FIGURE 10. Geometry in chart K_1 for $n = 2$.

Lemma 4.9. *There exists a set $U_1^{out} \subset \Sigma_1^{out}$ (specified in the proof below) such that for any $\tilde{P}_1^{out} \in U_1^{out}$, there is a unique $\tilde{P}_1 \in U_1 \subset \mathcal{V}_1$ with*

$$\tilde{P}_1^{out} \in \tilde{\Gamma}_1 \tag{99}$$

for the solution $\tilde{\Gamma}_1(\xi_1)$ of (95) starting in $\tilde{P}_1$; here, U_1 is an appropriately defined subset of $\mathcal{V}_1$ containing P_1.

Proof. Let $\tilde{P}^{out} := (\tilde{u}^{out}, \tilde{v}^{out}, \varepsilon\delta^{-1}, \delta)$, and let $U^{out} \in \Sigma^{out}$ be defined such that $|\tilde{u}^{out} - 1| \leq \beta$ for some $\beta > 0$ to be determined. From (97) we have

$$\delta = \varepsilon e^{\xi} \tag{100}$$

in Σ^{out}, whence

$$v_0 = \tilde{u}^{out} \left(\ln \frac{\delta}{\varepsilon} \right)^{-1} \in \left[(1-\beta) \left(\ln \frac{\delta}{\varepsilon} \right)^{-1}, (1+\beta) \left(\ln \frac{\delta}{\varepsilon} \right)^{-1} \right]; \tag{101}$$

here, β is chosen such that $(1+\beta)\big(\ln \frac{\delta}{\varepsilon_0}\big)^{-1} \leq \alpha$. The same is true of $\tilde{v}^{out}$, which together with $\varepsilon \in [0, \varepsilon_0]$ determines both U^{out} and U, see Figure 10. $\square$

Let us fix $\tilde{u}_1^{out} = 1$ in the definition of $\tilde{P}_1^{out}$ now, and take $\tilde{\Gamma}_1$ to be the corresponding solution to (95). Let $\tilde{P}_1^{in} := \tilde{\Gamma}_1 \cap \Sigma_1^{in}$, as before. With (95) being linear, its

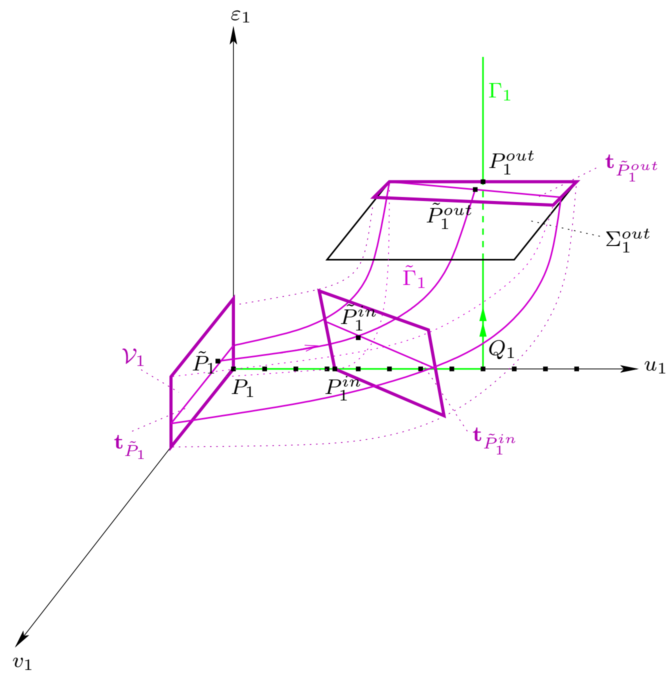

FIGURE 11. Dynamics in chart K_1 for $n = 2$.

variational equations are given by

$$\begin{aligned} \delta u_1' &= \delta v_1, \\ \delta v_1' &= 0, \\ \delta r_1' &= -\delta r_1, \\ \delta \varepsilon_1' &= \delta \varepsilon_1, \end{aligned} \tag{102}$$

which is again just (95). Let $\widetilde{\mathcal{M}}_1$ denote the saturation of $\mathcal{V}_1$ under the flow of (95); as for $n = 3$, we obtain the following

Proposition 4.10. *Let* $\mathbf{t}_{\tilde{\Gamma}_1} \in T_{\tilde{\Gamma}_1}\widetilde{\mathcal{M}}_1$ *be the solution of* (102) *corresponding to* $\tilde{\Gamma}_1(\xi_1)$. *Then,* $\mathbf{t}_{\tilde{P}_1^{in}} \in T_{\tilde{P}_1^{in}}\widetilde{\mathcal{M}}_1$ *is spanned by*

$$(\delta u_1, \delta v_1, \delta r_1, \delta \varepsilon_1)^T(-\ln \rho) = (-\ln \rho, 1, 0, 0)^T. \tag{103}$$

The above result not only implies that again $\mathbf{t}_{\tilde{P}_1^{in}}$ already is almost horizontal, but in fact even more so than for $n = 3$ (see Figure 11). We can now proceed by stating the analogues of Lemma 4.3 and Proposition 4.4 here:

Lemma 4.11. *For any* $u_{1_0}, v_{1_0} \geq 0$, *the solutions* $u_1(\xi_1)$ *and* $v_1(\xi_1)$ *to* (36) *can be estimated as follows:*

$$\begin{aligned} u_{1_0} \leq u_1(\xi_1) &\leq u_{1_0} + v_{1_0}\xi_1, \\ 0 \leq v_1(\xi_1) &\leq v_{1_0}. \end{aligned} \tag{104}$$

Proof. The proof is the same as for Lemma 4.3. $\square$

Proposition 4.12. *Let $\tilde{\Pi}: \Sigma_1^{in} \to \Sigma_1^{out}$ denote the transition map for* (95), *and let $\tilde{P}_1^{out}$ be defined as above. Then, $D\tilde{\Pi}(\mathbf{t}_{\tilde{P}_1^{in}}) = \mathbf{t}_{\tilde{P}_1^{out}}$ is spanned by*

$$(\delta u_1, \delta v_1, \delta r_1, \delta\varepsilon_1)^T\Big(\ln\frac{\delta}{\varepsilon}\Big) = \Big(\ln\frac{\delta}{\varepsilon}, 1, 0, 0\Big)^T. \tag{105}$$

Proof. Equations (102) can be solved explicitly: from Proposition 4.10, one easily obtains $\delta r \equiv 0 \equiv \delta\varepsilon$ and

$$(\delta u, \delta v)^T(\xi) = (\xi, 1)^T. \tag{106}$$

The assertion then follows by taking $\xi = \ln\frac{\delta}{\varepsilon}$. □

It only remains to show that for ε small, the solution of the full problem (36) starting in $\tilde{P}_1$ does indeed stay close to $\tilde{\Gamma}_1$:

Lemma 4.13. *Let $u_{1_0} = 0$, let v_{1_0} be defined as in Lemma 4.9, and let u_1^{out} and v_1^{out} denote the values of u_1 and v_1 in Σ_1^{out} for the corresponding solution to* (36). *Then,*

$$1 - \varepsilon\ln\frac{\delta}{\varepsilon} \le u_1^{out} \le 1, \tag{107a}$$

$$\Big(\ln\frac{\delta}{\varepsilon}\Big)^{-1} - \varepsilon \le v_1^{out} \le \Big(\ln\frac{\delta}{\varepsilon}\Big)^{-1}. \tag{107b}$$

Proof. For the proof, note first that

$$u(\xi) = u(\xi_0) + \int_{\xi_0}^{\xi} v(\xi')d\xi', \tag{108a}$$

$$v(\xi) = v(\xi_0) - \int_{\xi_0}^{\xi} \varepsilon(\xi')u(\xi')v(\xi')d\xi' \tag{108b}$$

for any $0 \le \xi_0 \le \xi \le \ln\frac{\delta}{\varepsilon}$. The upper bounds are obtained directly from Lemma 4.11 with $\xi_0 = 0$ and $\xi = \ln\frac{\delta}{\varepsilon}$; for the lower bounds, we rewrite (108b) as

$$\begin{aligned} v(\xi) &\ge v(\xi_0) - \varepsilon\int_{\xi_0}^{\xi} e^{\xi'}v(\xi')d\xi' \\ &= \big(1 - e^{\xi_0}\big)\,v(\xi_0) + e^{\xi_0}v(\xi_0) - \varepsilon\int_{\xi_0}^{\xi} e^{\xi'}v(\xi')d\xi'; \end{aligned} \tag{109}$$

here we have used that $\max_{\xi'\in[\xi_0,\xi]} u(\xi') \le 1$ and $\varepsilon(\xi) = \varepsilon e^{\xi}$. To complete the proof, we require the following generalization of *Gronwall's inequality*, see [Bee75] or [Gol69]:

Lemma 4.14. *Let the real-valued functions $y(t)$, $k(t)$ be continuous on $I := [\alpha,\beta] \subset \mathbb{R}$, and let the functions $b(t)$, $k(t)$ be non-negative on I. If $x(t)$ is any function such that*

$$x(t) \ge y(t_0) - b(t)\int_{t_0}^{t} k(\tau)y(\tau)d\tau, \qquad \alpha \le t_0 \le t \le \beta, \tag{110}$$

then

$$x(t) \ge y(t_0)\exp\Big[-b(t)\int_{t_0}^{t} k(\tau)d\tau\Big], \qquad \alpha \le t_0 \le t \le \beta. \tag{111}$$

This result is optimal in the sense that equality in (110) *implies equality in* (111).

Setting $y(t) = e^t v(t)$, $k(t) \equiv 1$, and $b(t) \equiv \varepsilon$, we obtain (with ξ instead of t and $\alpha = 0$, $\beta = \ln \frac{\delta}{\varepsilon}$)

$$v(\xi) \geq \left(1 - e^{\xi_0}\right) v(\xi_0) + e^{\xi_0} v(\xi_0) e^{-\varepsilon(\xi - \xi_0)}, \tag{112}$$

whence

$$v^{out} \geq v_0 e^{-\varepsilon \ln \frac{\delta}{\varepsilon}} \geq v_0 \left(1 - \varepsilon \ln \frac{\delta}{\varepsilon}\right). \tag{113}$$

The estimate for u^{out} now follows from (108a) and (113). □

***Remark* 10.** One easily sees that the above is equivalent to

$$u_1^{out} = 1 + \mathcal{O}(\varepsilon \ln \varepsilon), \qquad v_1^{out} = \mathcal{O}\big((\ln \varepsilon)^{-1}\big). \tag{114}$$

A similar result might generally be expected if Lemma 4.13 were to be rephrased in terms of $\Pi : \Sigma_1^{in} \to \Sigma_1^{out}$, the transition map for (36). □

4.2.2. *Dynamics in chart K_2.* In contrast to the situation in K_1, the dynamics in K_2 is not at all more involved for $n = 2$ than it is for $n = 3$. We will therefore not go into too many details here. Given Lemma 4.5, which is equally valid for $n = 2$, the variational equations along γ_2 are found to be

$$\begin{aligned} \delta u_2' &= \delta v_2, \\ \delta v_2' &= -\left(\frac{1}{\xi_2} + 1\right) \delta v_2, \\ \delta \eta_2' &= -\frac{2}{\xi_2} \delta \eta_2, \\ \delta r_2' &= 0, \end{aligned} \tag{115}$$

where γ_2 is defined as in (74). The solution to

$$\delta u_2'' = -\left(\frac{1}{\xi_2} + 1\right) \delta u_2' \tag{116}$$

now is given by

$$\delta u_2(\xi_2) = \alpha E_1(\xi_2) + \beta \tag{117}$$

with α, β constant. Just as for $n = 3$, we have the following result:

Proposition 4.15. *Let Q_2, $\mathcal{W}_2^s$, $\mathbf{t}_{\Gamma_2}$, and Q_2^{in} be defined as above. Then, $\mathbf{t}_{Q_2^{in}} \in T_{Q_2^{in}} \mathcal{W}_2^s$ is spanned by*

$$(\delta u_2, \delta v_2, \delta \eta_2, \delta r_2)^T(\delta) = \left(\delta u_2^{in}, 1, 0, 0\right)^T, \tag{118}$$

where

$$\delta u_2^{in} = \mathcal{O}(\delta \ln \delta). \tag{119}$$

Proof. The proof is exactly the same as for $n = 3$. □

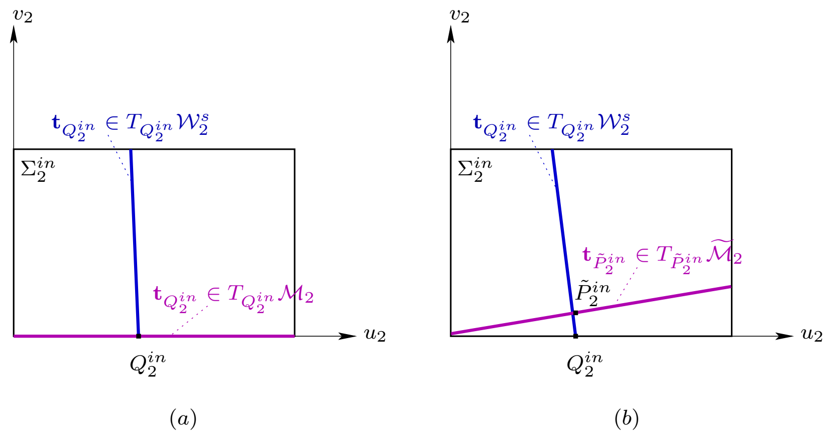

FIGURE 12. Illustration of transversality in Σ_2^{in} for (a) $n = 3$ and (b) $n = 2$.

5. Proof of Theorem 4.1

Having finished the preparatory work, we are now ready to prove our main result:

Proof of Theorem 4.1. As noted before, it suffices to prove existence and uniqueness for the blown-up system. For (16),(17) proper, the assertion then follows by applying the appropriate blow-down transformations.

For $n = 3$, a direct computation using Lemma 3.5 and Proposition 4.4 shows that $\mathbf{t}_{P_1^{out}}$ corresponds to the vector $\mathbf{t}_{Q_2^{in}} \in T_{Q_2^{in}}\mathcal{M}_2$ spanned by

$$(\delta u_2, \delta v_2, \delta \eta_2, \delta r_2)^T(\delta) = (1, 0, 0, 0)^T; \tag{120}$$

here, $Q_2^{in} = \kappa_{12}\left(P_1^{out}\right)$ and $\mathcal{M}_2 = \kappa_{12}\left(\mathcal{M}_1\right)$. With Proposition 4.7, this implies transversality in Σ_2^{in} for $r_2 = 0$, see Figure 12. From regular perturbation theory, Lemma 4.3, and the proof of Proposition 4.4 it now follows that $\mathcal{M}_2$ and $\mathcal{W}_2^s$ still intersect transversely for $r_2 > 0$ sufficiently small.

For $n = 2$, define $\tilde{P}_2^{in} := \kappa_{12}\big(\tilde{P}_1^{out}\big)$ and $T\kappa_{12}\big|_{\Sigma_1^{out}}\big(\mathbf{t}_{\tilde{P}_1^{out}}\big) =: \mathbf{t}_{\tilde{P}_2^{in}} \in T_{\tilde{P}_2^{in}}\widetilde{\mathcal{M}}_2$. By Proposition 4.12, $\mathbf{t}_{\tilde{P}_2^{in}} \cap \mathbf{t}_{Q_2^{in}}$ is then clearly transversal, and with Lemma 4.13 the intersection remains transversal for (95) replaced by (36) (see the proof of Proposition 4.4 again: the estimate in (69) is valid for $n = 2$, as well, as the relevant equation is

$$z' = 1 + \varepsilon^{in} e^{\xi} \tilde{u} z + \varepsilon^{in} e^{\xi} \tilde{v} z^2 \tag{121}$$

now). $\square$

***Remark* 11.** The above proof shows why it suffices to consider only one tangent direction each both in $T\mathcal{M}_2$ and in $T\mathcal{W}_2^s$: as the equations in K_2 are completely independent of $r_2 = \varepsilon$, the question of transversality is reduced to two (instead of three) dimensions in Σ_2^{in}. $\square$

The meaning of Theorem 4.1 is the following: for any value of $\varepsilon \in (0, \varepsilon_0]$, there is exactly one pair of values $(u_{2_\varepsilon}, v_{2_\varepsilon})$ singled out by the intersection of $\mathcal{M}_2$ and $\mathcal{W}_2^s$ in Σ_2^{in}. For $\varepsilon = 0$, of course, one again retrieves the singular orbits discussed

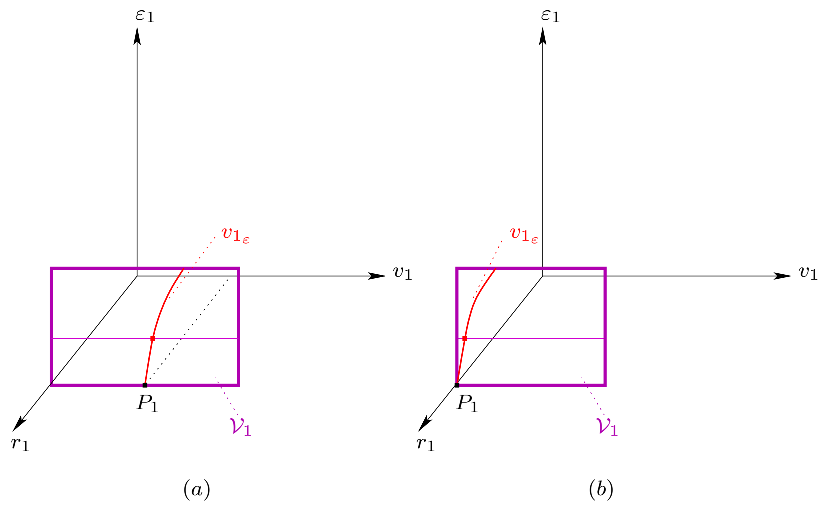

FIGURE 13. v_{1_ε} for (a) $n = 3$ and (b) $n = 2$.

above. These pairs $(u_{2_\varepsilon}, v_{2_\varepsilon})$ form a curve parametrized by $\varepsilon \in [0, \varepsilon_0]$ which, after transformation to K_1, determines a curve of boundary values in $\mathcal{V}_1$, $(0, v_{1_\varepsilon}, 1, \varepsilon)$, say.[8] It is precisely the function v_{1_ε} which, if explicitly known, would give us the solution to (29). For an illustration of the above argument, see Figure 13.
In the upcoming paper [PS], we will derive expansions for v_{1_ε} both for $n = 3$ and for $n = 2$; as is to be expected, these expansions agree with those obtained in the literature by asymptotic matching.

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[8]Note again that due to $r_1 = 1$ in $\mathcal{V}_1$, we can write ε instead of ε_1.

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E-mail address: nikola.popovic@tuwien.ac.at, peter.szmolyan@tuwien.ac.at

Department of Applied Mathematics and Numerical Analysis, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria, EU.