MA561 - Fall 2024 Homework 1 - Due Friday, Sept. 20th, 9am on Gradescope

Problem 1 *(Olver 1.1.5): Show that the following functions* $u(x,y)$ *define classical solutions to the two-dimensional Laplace equation* $u_{xx} + u_{yy} = 0$*. Be careful to specify an appropriate domain.*

(a) $e^x \cos y$, (b) $1 + x^2 - y^2$, (c) $x^3 - 3xy^2$, (d) $\log(x^2 + y^2)$, (e) $\frac{x}{x^2+y^2}$.

Problem 2 *(Olver 1.1.6) Find all solutions* $u = f(r)$ *of the two-dimensional Laplace equation* $u_{xx} + u_{yy} = 0$ *that depend only on the radial coordinate* $r = \sqrt{x^2 + y^2}$*. Hint: substitute the definition of* r *into* f *and insert into the equation (applying the chain-rule). You should obtain an ordinary differential equation for* f *in terms of* r *which you can then solve.*

Problem 3 *(Olver 1.1.9) Find all polynomial solutions* $p(x,t)$ *of the heat equation* $u_t = u_{xx}$ *with* $\deg p \leq 3$.

Problem 4 *(Olver 1.23) Show that, on* $\mathbb{R}^3$*, the gradient* $L_1[u] = \nabla u$*, curl* $L_2[U] = \nabla \times U$*, and divergence operator* $L_3[U] = \nabla \cdot U$*, all define linear differential operators. For the last two operators* U *denotes the vector valued function* $U = (u_1, u_2, u_3)^T$ *where* u_1, u_2*, and* u_3 *are all functions of three variables,* (x, y, z)*.*

Problem 5 *(Olver 1.20) The displacement* $u(x,t)$ *of a forced violin string is modeled by the partial differential equation* $u_{tt} = 4u_{xx} + F(x,t)$ *(we'll discuss this in a week or two). When the string is subjected to the external forcing* $F(x,t) = \cos x$*, verify* $u(x,t) = \cos(x - 2t) + \frac{1}{4}\cos x$ *is a solution, while when* $F(x,t) = \sin x$*, verify* $u(x,t) = \sin(x - 2t) + \frac{1}{4}\sin x$ *is a solution. Use these to find a solution when the forcing function* $F(x,t)$ *is (a)* $\cos x - 5\sin x$ *and (b)* $\sin(x - 3)$*.*

Problem 6 (Linear Cable Equation) *On either side of a biological cell membrane is an aqueous solution of ions. The ions' concentrations are usually different between the inside and outside of the cell, giving rise to an electrical potential and a membrane voltage,* V*. The membrane itself is impermeable. If the membrane has ion channels (pores created by transmembrane proteins), ions can cross the membrane and* V *is dynamic. The reversal potential of those ions,* E*, is the membrane potential that balances their concentration gradient so there is no net flux across the cell membrane. In a cylindrical portion of a cell with only (very many) passive channels, the membrane potential* $V(x,t)$ *obeys the linear cable equation*

$$\lambda^2 V_{xx} = \tau V_t + V - E,$$

where λ *and* τ *are space and time constants and* x *is the linear position (assuming* V *is radially symmetric). Assuming* E *is a real constant and* $(x,t) \in D = \mathbb{R} \times \mathbb{R}$*, state and justify two solutions to this equation. Hint: Look for solutions which are one of the following: (i) independent of both time and space, (ii) independent of space* x*, (iii) independent of time* t*.*

Problem 7 (Olver 1.25) Suppose L and M are linear differential operators, explain why the following are also linear differential operators:

(a) $L - M$

(b) $3L$

(c) fL where f is an arbitrary function of the independent variables

(d) $L \circ M$.