MA561 - Fall 2024 Homework 2 - Due Friday, Sept. 27th, 9am on Gradescope

Note, for all problems which ask you to graph the solution, you are welcome to embed figures made from a software package such as Mathematica or Matlab to more efficiently and accurately plot solution profiles you obtain.

Problem 1 *(Olver 2.1.3):* Find the general solution $u(x,t)$ to the following PDEs:

(a) $u_t = x - t$; (b) $u_x + tu = 0$;

Problem 2 *(Olver 2.2.2):* Solve the following initial value problem and graph the solutions at times $t = 1, 2, 3$:

$$u_t - 4u_x + u = 0, \ u(x,0) = 1/(1+x^2).$$

Problem 3 *(Olver 2.2.4):* Solve the initial value problem $u_t + 2u_x = 1, \quad u(x,0) = e^{-x^2}$. (Hint: use characteristic coordinates to transform the PDE into a simpler equation).

Problem 4 *(Olver 2.2.17):* (a) Find and sketch the characteristic curves.

(b) Solve the initial value problem $u_t - xu_x = 0, \quad u(x,0) = (x^2+1)^{-1}$.

(c) Graph the solution at times $t = 0, 1, 2, 3$.

(d) What is $\lim_{t\to+\infty} u(x,t)$ for each x?

For the Problem 5 and 6 equations consider the nonlinear transport equation

$$u_t + uu_x = 0. \tag{0.1}$$

Problem 5 *(Olver 2.3.1):* Discuss the behavior of the solution to equation (0.1) for the following initial data.

(a) $u(x,0) = \begin{cases} -2, & x < -1; \\ 1, & x > -1; \end{cases}$ (b) $u(x,0) = \begin{cases} 1, & x < 1; \\ -2, & x > 1; \end{cases}$

Problem 6 *(Olver 2.3.3 + 2.3.4):* (a) Let $u(x,0) = (1+x^2)^{-1}$. Does the resulting solution of (0.1) produce a shock wave at some finite positive time? If so, find the earliest shock time and sketch (without explicitly solving) a graph of the solution just before and soon after the shock wave. If not, explain what happens to the solution as t increases.

(b) Answer the same questions for $u(x,0) = x(x^2+1)^{-1}$.

Problem 7 *(Olver 2.3.2):* Solve the initial value problem

$$u_t - uu_x = 0, \qquad u(x,1) = \begin{cases} -1, & x < 0 \\ 3, & x > 0 \end{cases}$$