MA561 - Fall 2024 Homework 3 - Due Friday, Oct. 4th, 9am on Gradescope

Problem 1 *(Traffic Modeling)*: (see texts by Haberman or Logan for example) Consider cars travelling down a 1-dimensional road with no entrances or exits. Let $u(x,t)$ denote the traffic density (number of cars per mile) at time t located at road position x. Let the traffic flow (or flux) $\phi(x,t)$ denote the number of cars per hour passing through x. Using techniques described in class the following conservation law can be derived for u

$$u_t + \phi_x = 0$$

(a) Let $v(x,t)$ be the velocity of cars at (x,t), give a brief explanation why the traffic flow satisfies $\phi = uv$.

(b) Assume i) the velocity only depends on car density $v = v(u)$, ii) at some fixed maximum density $u_{max} > 0$ the velocity is 0, and iii) when car density is zero cars travel at a fixed maximum velocity $v_{max} > 0$. That is $v(u_{max}) = 0$ and $v(0) = v_{max}$. Propose a simple expression for $v(u)$ which satisfies these assumptions.

(c) Write down the resulting partial differential equation for your velocity function (and hence traffic flow function) and give the equation for characteristic curves (using the methods discussed in class).

(d) (Red light turning green) Let $u(x,0) = \begin{cases} u_{max}, & x < 0 \\ 0, & x > 0 \end{cases}$, which represents cars starting behind a red light at $x = 0$ which then turns green at $t = 0$. Solve the initial value problem and describe the behavior of the cars for $t > 0$.

(e) (green light turning red) Assume for $t < 0$ a traffic light, located at $x = 0$, controlling the flow is green, traffic density is constant $u \equiv u_0$ and all cars move at the same velocity. Now say the light, at $x = 0$, turns red at $t = 0$, so behind the light, $x < 0$, the density is still uniform $u(x,0) = u_0$. At the light, the density is maximal $u(0,t) = u_{max}$ for all times $t \geq 0$. This creates a shock at the point $(x,t) = (0,0)$ which evolves backwards into the region $x < 0$, describe how the shock evolves as $t > 0$ increases. (Hint: only consider the domain $\{x \leq 0\}$ and use the Rankine-Hugoniot condition for general ϕ on page 47 of Olver's text).

Problem 2 *(Olver 2.3.10)* An N-wave is a solution to the initial value problem

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} mx, & -\ell \leq x \leq \ell \\ 0, & \text{otherwise} \end{cases}$$

for $m, \ell > 0$. Give the solution formula when $m, \ell > 0$ and discuss how the solution evolves if $m < 0$.

Problem 3 *(Olver 2.4.1 + 2.4.2)*: Consider the wave equation $u_{tt} = u_{xx}$. For each of the initial conditions, give the solution (in as simple a form as possible) and sketch/plot it at several representative times.

(a) $u(x,0) = e^{-x^2}, \quad u_t(x,0) = \sin(x)$.

(b) $u(x,0) = \begin{cases} 1, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}, \qquad u_t(x,0) = 0.$