MA561 - Fall 2024 Homework 5 - Due Friday, Nov. 1, 5pm on Gradescope

Problem 1 *Olver 4.1.1:* Suppose the ends of a bar of length 1 and thermal diffusivity $\gamma = 1$ are held fixed at respective temperatures $u(0,t) = 0$ and $u(1,t) = 10$.

(a) Determine the equilibrium temperature profile.

(b) Determine the rate at which the equilibrium temperature profile is approached.

(c) What does the temperature profile look like as it nears equilibrium?

Problem 2 *(Olver 4.1.7):* Consider the same bar, but this time assume it is thermally insulated, so that $u_x(0,t) = 0 = u_x(1,t)$ and has the initial temperature distribution $u(x,0) = \begin{cases} x, & 0 < x \leq 1/2 \\ 1-x, & 1/2 \leq x < 1 \end{cases}$.

(a) Use Fourier Series (that is eigensolution decomposition) to write down the temperature distribution at time $t > 0$.

(b) What is the equilibrium temperature distribution for the bar (i.e. what happens as $t \to +\infty$)?

(c) How fast does the solution go to equilibrium? Sketch solutions and discuss their behavior.

Problem 3 *(Olver 4.1.13 and 4.1.14):* (a) Explain why the thermal energy $E(t) = \int_0^\ell u(x,t)dx$ is not constant for the heat equation on the interval $x \in [0,\ell]$ with homogeneous Dirichlet boundary conditions and any initial value problem with $u(x,0)$ not identically equal to zero (i.e. $u(x,0) \neq 0$).

(b) Now show that $E(t)$ is constant if instead the equation has homogeneous Neumann boundary conditions on $x \in [0,\ell]$.

Hints: Compute $E'(t)$, use the fact that $u(x,t)$ solves the heat equation and integrate by parts might. Alternatively, plugging in the Fourier series representation of the solution might help.

Problem 4 *(Olver 4.2.1 and 4.2.2):* In music, an octave corresponds to doubling the frequency of the sound waves. Say on a piano the middle C string has length 0.7 meters while the string for the C an octave higher has length 0.6 meter.

(a) Assuming that they have the same density, how much tighter does the shorter string need to be tuned?

(b) How much longer would a piano string have to be to make the same sound when it is pulled twice as tight.

Hint: Recall that in the wave equation, the wave speed has $c^2 = T_0/\rho_0$ where ρ_0 is the density and T_0 is the tension of the string, recall how separation of variables was used to find the temporal oscillation frequencies of sinusoidal modes for homogeneous Dirchlet boundary conditions).

Problem 5 *(Olver 4.2.3):*.

Write down the solution to the following initial boundary value problems for the wave equation $u_{tt} = 2u_{xx}$ with the following boundary and initial conditions: $u(0,t) = 0 = u(\pi,t)$, $\qquad u(x,0) = 0, u_t(x,0) = 1$.

Problem 6 *(Olver 4.2.35):* Let $u(x,t)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$ on $0 < x < \ell$, satisfying homogeneous Dirichlet boundary conditions. The total energy of u at time t is

$$E(t) = \int_0^\ell \frac{1}{2}\left[(u_t)^2 + c^2(u_x)^2\right] dx.$$

Show that energy is conserved, that is $E(t) = E(0)$ for all time $t \geq 0$.