MA561 - Fall 2024 Homework 7 - Due Friday, Nov. 15, 5pm on Gradescope

This homework is a little different. For each problem, include both text describing your findings and some figures, plots, or tables to describe how solutions behave. Also include the code you used in the PDF after each problem. If you are having trouble coding anything or getting any numerical software running please come see me as soon as possible. Feel free to use and alter the code framework which I showed in class and posted in blackboard. Matlab live notebooks, Mathematica Notebooks, or Jupyter (Python) Notebooks might be useful in presenting your work/code/solutions.

Problem 1 *Olver 5.2.5*: (Heat equation with time dependent Dirichlet condition) Numerically solve the following initial value problem with time-depdendent boundary condition which represents a varying temperature at one end of a bar.

$$\begin{aligned} &u_t = u_{xx}, \qquad x \in (0, 10) \\ &u(x, 0) = 0, \quad u(0, t) = \sin(t), \quad u(10, t) = 0. \end{aligned}$$

Use space step sizes $\Delta x = 0.1, 0.05$ with suitably chosen time steps Δt for each, and both the explicit (forward) and implicit (backward) time discretization and second order finite differences in space.

Problem 2 *Olver 5.1.5*: You are asked to derive some basic one-sided finite difference formulas used for approximating derivatives of functions at or near boundaries.

(a) Construct a finite difference formula that approximates the derivative $u'(x)$ using the values $u(x), u(x+h), u(x+2h)$ (hint: Taylor expand each of these and take linear combinations of them to approximate $u'(x)$). What is the order of your formula?

(b) Test your formula by computing the approximations to the first derivative of $u(x) = e^{x^2}$ at $x = 1$ using step sizes $h = 0.1, 0.01, 0.001$. What is the error in your numerical approximations? Are the errors compatible with the theoretical orders your computed? Discuss why or why not.

Problem 3 *Olver 5.2.6*: Describe in detail how you would modify the implicit scheme when the heat boundary value problem has Neumann boundary conditions $u_x(0, t) = 0 = u_x(\ell, t)$. Be sure to give a full description of the numerical scheme you would use in each case.

(b) Test this on the boundary value problem with $u(x, 0) = 1/2 + \cos(2\pi x) - \frac{1}{2}\cos(3\pi x)$, $u_x(0, t) = 0 = u_x(1, t)$ using step sizes $\Delta x = 0.1, = 0.01$ and appropriate time steps. Compare your numerical solution with the exact solution at times $t = 0.01, 0.03, 0.05$ (hint for the exact solution: note this will be a cosine series with only 3 terms, what are they?) and try to explain any discrepancies.

Problem 4 *Olver 5.3.7:* Nonlinear transport equations are often solved numerically by writing them in the form of a conservation law, and then applying finite difference formulas directly to the conserved density and flux.

(a) Devise an upwind scheme for numerically solving our favorite nonlinear transport equation $u_t + \frac{1}{2}(u^2)_x = 0$ with non-negative initial data (so that characteristic lines all move rightward).

(b) Test your scheme on the initial value problem $u(x, 0) = e^{-x^2}$. Describe what happens as the solution approaches the shock time $t_* = \sqrt{e/2}$. (this was computed using the shock time formula from chapter 2).