MA561 - Fall 2024 Homework 8 - Due Tuesday, Nov. 26th, 5pm on Gradescope

Problem 1 *Olver 5.3.3*: Consider the initial value problem

$$u_t + \frac{3x}{x^2+1} u_x = 0, \qquad u(x,0) = (1 - \frac{x^2}{2})e^{-x^2/3}.$$

On the interval $[-5, 5]$ using space step size $\Delta x = 0.1$, time step size $\Delta t = 0.025$, and final time $t = 1.5$ apply

(a) The right-sided finite difference scheme (eqn. (5.38) in text but modified for variable wave speed $c(x)$ - that is at each spatial grid point m, set $c = c(x_m)$)

(b) The left-sided finite difference scheme (eqn. (5.44) again modified for variable wave speed $c(x)$)

(c) The up-wind scheme (eqn. (5.49) in the text)

Graph the resulting numerical solutions at times $t = 0.5, 1, 1.5$ for each and discuss what you observe in each case. Which of the schemes are stable?

Problem 2 *(Jump rope attached to the wall)* Now consider the wave equation $u_{tt} = u_{xx}$ on the domain $x = [0, 5]$ with trivial initial conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$, one oscillating boundary condition $u(0, t) = \sin(\omega t)$ and one pinned boundary condition $u(5, t) = 0$ representing a string being whipped back and forth on the left side.

(a) Set $\Delta x = 0.01$, what range of Δt values are allowed?

(b) Test your answer by implementing the numerical solution with $\omega = \pi$ and time ranging from $t = 0$ to $t = 20$ for one value of Δt in the allowable CFL range and one value outside. Discuss what you observe in your numerical solutions.

(c) Choosing a Δt solidly in the stable range, investigate how the solution behaves as ω is varied. Discuss how the frequency of the observed solution changes. Trying three to four ω values will suffice.

Problem 3 Solve the Dirichlet problem from Problem 3 of homework 6:

$$\Delta u = 0, \qquad u(x, 0) = 0, u(x, \pi) = 0, u(0, y) = \sin(y), \quad u(\pi, y) = 0.$$

numerically using the finite difference scheme discussed in class (and in the book) setting $\Delta x = \Delta y$. Compare your numerical solution with the true solution found previously $u(x, y) = \frac{\sin(y)\sinh(\pi - x)}{\sinh(\pi)}$ and discuss how the error converges as $\Delta x = \Delta y$ is halved multiple times (i.e. how does the error reduce when setting$\Delta x \mapsto \Delta x/2$?). Coding Notes:

- Since π is irrational, one can define an evenly spaced grid in $x \in [0, \pi]$ (and hence also in y) using the matlab command "$xb = \text{linspace}(0, \pi, n_x + 2)$" which gives a grid from 0 to π using $n_x + 2$ points (including boundaries points $x = 0, x = \pi$) with spacing $\Delta x = \pi/(n_x + 1)$. You can roughly halve the stepsize by doubling n_x.
- For plotting purposes, it's helpful to rearrange your solution vector U into a matrix V with entries $v_{i,j}$ aligned with the x, y grid. The command "reshape" may be helpful going between this and the solution vector U.