Evans Problems

Chapter 5: Problems 14,17,20,21

Remark: Problem 21 shows which spaces $H^s(\mathbb{R}^n)$ are Banach algebras, a useful characteristic when studying nonlinear PDE with polynomial nonlinearities.

Additional Problems :

Problem 1: Let $\Omega = (-1,1) \subset \mathbb{R}$, and $u(x) = |x|$. Show that $u \in W^{1,p}(\Omega)$ but $u \not\in W^{2,p}(\Omega)$ for $p \in (1, \infty)$.

Problem 2: (a): (Ehrling’s Lemma): Let $X,Y,Z$ be Banach spaces such that $X$ is compactly embedded in $Y$ and $Y$ is continuously embedded in $Z$. Prove that for every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$
\|u\|_Y \leq \epsilon \|u\|_X + C(\epsilon) \|u\|_Z,$$

for all $u \in X$. (Hint: go by contradiction with a sequence $\{u_n\} \in X$ with $\|u_n\|_X = 1$)

(b): Let $\Omega$ be bounded with $C^1$ boundary. Apply part (a) to the spaces $X = H^k(\Omega), Y = H^{k-1}(\Omega), Z = L^1(\Omega)$ and show that the standard norm $\|u\|_{H^k}^2 = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2}^2$ is equivalent to the following norm

$$
\|u\|_{H^{k,*}}^2 := \sum_{|\alpha| = k} \|D^\alpha u\|_{L^2}^2 + \|u\|_{L^1}^2.
$$

Problem 3: Let $u,v \in H^1(\mathbb{R})$. Prove that

$$
\int_{-\infty}^{\infty} u(x)v'(x)dx = -\int_{-\infty}^{\infty} u'(x)v(x)dx.
$$

Problem 4: Let $\Omega$ be open, bounded with $C^1$ boundary. Show that every weakly convergent sequence in $H^1(\Omega)$ converges strongly in $L^2(\Omega)$. (See the appendices for more on weak convergence).