MA771 - Fall 2020 Homework 4 - Due Wednesday, Nov. 11

Please scan or type your homework and email it to me by the due date. Homeworks will be graded electronically and emailed back to you.

Problem 1: Find a horseshoe in the cat map. Draw some a picture!

Problem 2: Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y) = (x/2\,,\, 2y - \frac{15}{8}x^3)$. This map has one fixed point at $(0, 0)$. Classify it and sketch the "phase portrait" (i.e. draw any stable/unstable invariant manifolds and how orbits behave near the fixed point) of this mapping. To do this find an explicit homeomorphism which conjugates the nonlinear map to its linearization at the fixed point (hint: it should be a polynomial transformation). Note, via this transform, you should be able to explicitly describe the stable and unstable manifolds of F.

Problem 3: (strong unstable manifolds) Let $T = diag(\lambda_{uu}, \lambda_u)$, the diagonal matrix with $\lambda_{uu} > \lambda_u > 1$ and consider a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ as in class (that is there exists $\epsilon > 0$ such that f is Lipshitz, with $f(0) = 0$ and $Lip(f - T) < \epsilon$. The aim of this problem is to use the graph transform to construct a one-dimensional strong unstable manifold tangent to the strong-unstable eigenspace:

(i) First define what the graph transform should be in this case.

(ii) Next explain why, with the supremum norm considered in class, this will not give a contraction.

(iii) Next show that you can obtain a contraction when considering graphs σ with $\sigma(0) = 0$ using the projective metric

$$d(\sigma_1, \sigma_2) = \sup_{|x_{uu}|<1} \left| \frac{\sigma_1(x_{uu}) - \sigma_2(x_{uu})}{x_{uu}} \right|.$$

To start this, show that the set $\Sigma := \{\sigma : E^{uu} \to E^u \, : \, d(0, \sigma) \leq 1\}$ is a closed subset of the set of continuous graphs and that the linear map induces a graph transform which is a contraction in this metric.