MA771 - Fall 2020 Homework 5 - Due Wednesday, Dec. 9th

Please scan or type your homework and email it to me by the due date. Homeworks will be graded electronically and emailed back to you.

Barreira and Valls problems: Chapter 8: 13, 14 (try to give an example!), 20 (part of this is related to the example we worked out in class)

Additional Problems:

Problem 1: Show that the Lebesgue measure is invariant and ergodic for the cat map.

Problem 2: (Collet & Eckmann): Let $u(x) = \sin(2\pi x)$ and $f = E_2 : S^1 \to S^1$. This problem constructs an observable function in $L^2(S^1)$ with very slow decay of correlations.

(a): First show that $\int_0^1 u(x)dx = 0$ and $\int_0^1 u(x)u(f^n(x))dx = 0$ for any $n > 0$.

(b): Let $\{\alpha_k\}$ be the sequence defined as

$$\alpha_k = \begin{cases} 1/p, & k = 2^p \\ 0, & \text{otherwise} \end{cases}$$

and let $g(x) = \sum_k \alpha_k u(f^k(x))$. Show that there is a constant $C > 0$ such that for infinitely many vales of j such that $C_{g,g}(j) \geq \frac{C}{(\log j)^2}$.

(c): Finally, show that g is continuous (if you have time, observe/show that it is not differentiable).

Problem 3: Compute ergodic averages of $u(x, y) = \sin(2\pi x)$ numerically with several initial conditions, chosen at random, for the cat map. Can you measure or extract a rate of convergence? Try to interpret probabilistically what this means.

Optional Problem: (a) Prove the following statement: If for every continuous function ϕ chosen from a dense set Φ in the space of continuous functions, $C(X)$, the time averages $(\frac{1}{n}\sum_{k=0}^{n-1}\phi(f^k(x)))$ converge uniformly to a constant, then the mapping f is uniquely ergodic. Hint: First show that ergodic averages converge to a constant for every continuous function. Then consider what this means for any f-invariant probability measure.

(b): Use this result to prove that any rigid irrational circle rotation R_α is uniquely ergodic.