MA775 - Fall 2017 Homework 2 - Due October 3rd

Problem 1: Consider the ODE

$$\begin{aligned} x' &= -x + xy, \\ y' &= x - x^3 \end{aligned} \tag{0.1}$$

Find an Euler multiplier and determine the phase portrait (be careful with the sign for $x > 0$ and $x < 0$). Note here that x factors out of the vector field on the right hand-side. Locate periodic orbits, equilibria, and determine the omega-limit set for all initial conditions. Note it is alright to characterize some trajectories using an implicit equation (i.e. something like $f(x, y) = c$) and just draw their shapes.

Problem 2: Set $x' = Ax$, with $A = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$ and $x \in \mathbb{R}^2$, with $a, b > 0$.

(a)Derive an equation for the "projectivized flow". That is write $x = \theta|x|$ and derive an equation for $\theta \in S^1$.

(b)What are the equilibria of this flow? What are their eigenvalues and stability properties?

(c): Do the same for $A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$ for all $\mu \in \mathbb{R}$. Discuss how this approach helps qualitatively "unfold" or determine the dynamics near $\mu = 0$.

The linear flow induces a flow on the space of linear subspaces, i.e. the Grassmanian (useful in many settings such as heteroclinic bifurcation theory and geometric singular perturbation theory).

Problem 3_(Perko 1.8.6(3))_: Find the Jordan canonical form of the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Problem 4_(From Prof. Scheel's class)_: Consider an ODE $u' = f(u)$ with flow Φ_t and suppose $u = 0$ is an asymptotically stable equilibrium. Define the basin of attachtion

$$B = \{u_0; \lim_{t\to\infty} \Phi_t(u_0) = 0\}.$$

(a) Show that B is open and invariant.

(b) Show that the basin boundary ∂B is invariant.

(c): Define $\delta := dist(0, \partial B) = \inf_{y\in\partial B} |y|$ and explain why δ measures robustness of the equilibrium with respect to noisy time-dependent perturbations. Might be useful to consider a simple ODE such as $x' = x - x^3$ and the basins of the equilibria $x = \pm 1$.

These sets can in general be be incredibly complicated (i.e. look up riddled basin boundaries).