MA775 - Fall 2017 Homework 3 - Due October 24th

Problem 1*(Perko pg. 251)* Let

$$x' = -y + x\left((x^2+y^2)^2 - 3(x^2+y^2) + 1\right)$$
$$y' = x + y\left((x^2+y^2)^2 - 3(x^2+y^2) + 1\right)$$

Show there exists one periodic orbit in each annuli $A_2 = \{\mathbf{x} \in \mathbb{R}^2 \,:\, 1 < |\mathbf{x}| < 2\}$ and $A_1 = \{\mathbf{x} \in \mathbb{R}^2 \,:\, 0 < |\mathbf{x}| < 1\}$ and determine the stability properties of the periodic orbits. Note, polar coordinates may help.

Problem 2*(Chicone Exc. 1.129*: Construct an example of a differential equation, defined on all of $\mathbb{R}^3$ that has a periodic orbit, but no rest points.

Problem 3*(Prof. Scheel's Class)* Let

$$x' = -x + ay^2, \quad y' = -2y + bx^2.$$

(a) Draw the phase portrait of the linearization about the origin, and express trajectories as graphs $x = h(y)$ or $y = h(x)$.

(b) Find the Taylor expansion of the (smooth) strong stable manifold $x = h_{ss}(y)$ for the nonlinear system up to order 2.

(c) Try to find a one-dimensional "weak-stable" manifold tangent to $\{y = 0\}$, corresponding to the weak-stable eigenvalue $\lambda = -1$, by calculating a quadratic taylor expansion. What goes wrong?

(d) Set $a = 0$ and compute solutions explicitly and show there exist manifolds $y = h_s(x)$, but none is C^2 because of terms of the form $x^2 logx$.

Problem 4: Let

$$x' = \mu - x^2 + 2y^2, \quad y' = -y + x + \mu - x^2,$$

Study the parameter dependent center manifold at the equilibrium $(x, y; \mu) = (0, 0, \mu)$ for $\mu \sim 0$. In particular add an additional equation, $\mu' = 0$, and apply the center manifold theorem to obtain a second-order expansion of the center manifold. Calculate the leading order (i.e. the 2nd-order truncation) equation on the center manifold, and determine its qualitative dynamics for $\mu \sim 0$.