MA775 - Fall 2017 Homework 4 - Due November 14th

Problem 1*(Prof. Scheel's Class)* Let $u' = f(u, \mu)$, with $\mu \in \mathbb{R}, u \in \mathbb{R}^n$ have a periodic orbit $\gamma_*(t)$ with period $T > 0$ at $\mu = 0$ with Floquet Multiplier $\rho = 1$, algebraically simple. Prove that this orbit is robust in μ, that is it persists for all μ near 0. More precisely, prove that there exists a family of periodics $u_*(t; \mu)$ with period $T(\mu)$ smooth in μ.

Problem 2*(Arnold, Mathemtaical methods of classical mechanics)*: Let $u'' + a^2(t)u = 0$ with $a(t) = \omega + \epsilon$ for $t \in [0, \pi)$ and $a(t) = \omega - \epsilon$ for $t \in [\pi, 2\pi)$. Find the region in the (ϵ, ω) -plane for which the trivial state is stable using the following steps:

(a) Compute the period map $\Psi = \Phi_{2\pi,0}$.

(b) Show that the system is stable for $\text{tr}(\Psi) < 2$.

(c) Conclude stability criteria and formulas for boundaries of this region.

(c) Find expansions for these boundaries, $\omega(\epsilon)$ near $\epsilon = 0$ and $\omega \in \mathbb{N}/2$.

Problem 3*(Prof. Scheel's Class)* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-field whose associated ODE has a periodic orbit in a region Ω such that $\text{div} f < 0$. For $n = 2$ show that the periodic orbit is stable. Is this true for $n = 3$? Now assume $\text{div} f > 0$, can the periodic orbit be stable in any dimension n?

Problem 4*(Guckenheimer & Holmes 4.3.2*: Study the following systems using averaging methods:

$$x' = \epsilon(x - x^2)\sin^2 t, \qquad x' = \epsilon(x \sin^2 t - x^2/2).$$

Optional exercise for more practice explicitly calculating Floquet multipliers*(Chicone Exc. 2.91)*

Find a periodic solution of the system

$$\begin{aligned} x' &= x - y - x(x^2 + y^2), \\ y' &= x + y - y(x^2 + y^2), \\ z' &= -z. \end{aligned}$$

and determine its stability type by computing the Floquet multipliers of the period map.