MA775 - Fall 2017 Homework 5 - Due December 7th

Problem 1 *(Chicone Ex. 1.76)* Show that gradient systems can't have periodic orbits.

Problem 2 *(Degenerate Pitchfork, Prof. Scheel's class)*: Determine the bifurcation diagram of the equation:

$$x' = \mu_1 x + \mu_2 x^3 - x^5 + x\mathcal{O}(x^6 + |\mu| x^4).$$

That is, for all $(\mu_1, \mu_2) \in \mathbb{R}^2$, find all equilibria determine their stability.

Problem 3 *(Hopf persistence proof)[Iooss& Adelmayer]* As in class, consider $x' = F(z, \mu)$, $x \in \mathbb{R}^n, \mu \in \mathbb{R}, F(0,0) = 0$ such that $L := D_x F(0,0)$ has simple eigenvalues $\pm \mathrm{i}\omega, \omega > 0$ with eigenvectors $e_0, \overline{e_0}$ and with $\mathrm{spec}(L) \cap \mathrm{i}\mathbb{R} = \{\pm \mathrm{i}\omega\}$. As discussed in class, a center manifold reduction on the center subspace $E_0 = \{ze_0 + \overline{ze_0}\}_{z \in \mathbb{C}}$, along with a normal form reduction, gives the reduced system

$$\begin{aligned} z' &= \mathrm{i}\omega z + z g(|z|^2, \mu) + o(|A|^{2k+2}), \\ \bar{z}' &= -\mathrm{i}\omega \bar{z} + \bar{z}\overline{g(|z|^2, \mu)} + o(|z|^{2k+2}), \end{aligned}$$

with g a complex polynomial of order less or than equal to k in $|z|^2$, such that $g(0,0) = 0$. Introducing polar coordinates $z = r\mathrm{e}^{\mathrm{i}\phi t}$ we find

$$\begin{aligned} r' &= r g_r(r^2, \mu) + R_r \\ \phi' &= \omega + g_\phi(r^2, \mu) + R_\phi \end{aligned}$$

with $g_r = (g + \bar{g})/2 := \sum_{r=0}^{k} a_j(\mu) r^{2j}, \quad g_\phi = (g - \bar{g})/2 := \sum_{r=0}^{k} b_j(\mu) r^{2j}$ and $R_r = o(r^{2k+2}), R_\phi = o(r^{2k+1})$.

(a): Show that the system, truncated at order $2k+1$, undergoes a pitchfork bifurcation with direction determined by the leading order coefficients in g_r. You should find for $a := \frac{a_0'(0)}{a_1(0)} < 0$ the bifurcating solution takes the form, for $\mu > 0$,

$$r_*(\mu) = \sqrt{-a\mu} + o(|\mu|)), \quad \omega_*(\mu) = \omega + (b_0'(0) - b_1(0)a)\mu + o(|\mu|).$$

(b): Show that, for all (r, ϕ, μ) near zero, any solution must have $\frac{d\phi}{dt} > 0$. Use this to eliminate time, considering r as a function of ϕ, and derive a new ODE

$$\frac{dr}{d\phi} = \tilde{f}(r, \phi, \mu)$$

with $\tilde{f}$, 2π-periodic in ϕ.

(c): Let $r(0, \rho, \mu) = \rho$ be the initial data of solutions to this new ODE. Define a Poincare map, $(\rho, \mu) \mapsto P(\rho, \mu) := r(2\pi; \rho, \mu)$. Show that P can be factored $P(\rho, \mu) = \rho \tilde{P}(\rho, \mu)$ with smooth $\tilde{P}$ near $\rho, \mu \sim 0$ so that non-trivial fixed points of P are found by solving $\tilde{P}(\rho, \mu) = 1$.

(d): Show that $\tilde{P}(0,0) = 1$ and use the implicit function theorem to solve $0 = \tilde{F}(\rho, \mu) := \tilde{P}(\rho, \mu) - 1$ for μ (in terms of ρ) near $(0,0)$. Conclude the existence of a periodic orbit.

Hint: show that the function $\phi \mapsto \frac{\partial r}{\partial \rho}(\phi; 0, \mu)$ satisfies the ODE $\partial_\phi(\frac{\partial r}{\partial \rho}(\phi; 0, \mu)) = \frac{a_0(\mu)}{\omega + b_0(\mu)} \frac{\partial r}{\partial \rho}(\phi; 0, \mu)$ and conclude that $\partial_\rho r(\phi; 0, \mu) = \mathrm{e}^{(\frac{a_0(\mu)}{\omega + b_0(\mu)})\phi}$.

Problem 4 *(Bogdanov-Takens Normal form/The double-zero eigenvalue*

Consider the following system near $\underline{x} := (x, y) = (0, 0)$, with small parameters $\mu \sim 0$:

$$\dot{x} = y + \mathcal{O}(|\mu| + |\underline{x}|^2)$$
$$\dot{y} = 0 + \mathcal{O}(|\mu| + |\underline{x}|^2). \quad ,$$

(a): First study the normal form, up to second order, for $\mu = 0$, showing the above system can be transformed into the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^2 + xy + \mathcal{O}(|\underline{x}|^3). \quad , \end{aligned} \tag{0.1}$$ e:sys1

(Hint: One can proceed systematically as in [Guckenheimer & Holmes §3.3], studying the range of the homological operator.)

(b): (Unfolding) It can be found that the parameter dependent unfolding

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y - x^2 + xy + \mathcal{O}(|\underline{x}|^3), \quad \mu_1, \mu_2 \in \mathbb{R}, \end{aligned}$$

is a *versal deformation* of the vector-field. That is there exists a C^0-conjugacy between the linear part of this system and any small perturbation of the system (0.1) in a neighborhood of the origin (See [Arnold, Geometric Methods...] for more detail). Omitting the $\mathcal{O}(3)$-terms, study the stability properties of the equilibria $(x, y) = (0, \pm\sqrt{\mu_1})$, in the parameter plane $(\mu_1, \mu_2) \in \mathbb{R}^2$ and locate any curves in this plane where bifurcations occur. As much as you can, sketch the phase-portraits of the vector fields in each of the main parameter regions.

(c): There exists another bifurcation curve corresponding to a global bifurcation where the limit cycle vanishes along a homoclinic orbit. To study this, we wish to scale around this area: $\mu_1 \sim \mu_2^2$. That is scale as following: $x = \tilde{x}\epsilon^2, \quad y = \tilde{y}\epsilon^3, \quad \mu_2 = -\epsilon^2\nu_2, \mu_1 = \nu\epsilon^4, \frac{d}{d\tau} = \frac{d}{dt}\epsilon$, so that

$$\begin{aligned} \tilde{x}' &= y, \\ \tilde{y}' &= \nu - \epsilon\nu_2\tilde{y} + \epsilon\tilde{x}\tilde{y} - \tilde{x}^2. \end{aligned} \tag{0.2}$$

For $\epsilon = 0$ one obtains the "anharmonic" oscillator (dropping tilde's)

$$\begin{aligned} x' &= y, \\ y' &= \nu - x^2, \end{aligned} \tag{0.3}$$

Find the Hamiltonian for this system. For $\nu = 1$, the system has a homoclinic orbit explicitly given by

$$(x(t), y(t)) = (3\mathrm{sech}^2(t/\sqrt{2}) - 1, -3\sqrt{2}\mathrm{sech}^2(t/\sqrt{2})\tanh(t/\sqrt{2})).$$

Use the method of Melnikov to approximately find the parameter curve for which the homoclinic persists under the perturbation $\epsilon(0, -\nu_2\tilde{y} + \tilde{x}\tilde{y})$. Hint: One should approximately find $\nu_2 = 7/5$ and thus $\mu_1 = (5\mu_2/7)^2$ for $\mu_2 < 0$.

See [Guckenheimer & Holmes], [Kuznetsov], [Takens 1974], or [Bogdanov 1975] for more detail.