MA775 - Fall 2017 Homework 2 - Due October 6th

Problem 1: Consider the ODE

$$x' = -x + axy,$$
$$y' = x - x^2 \tag{0.1}$$

Find an Euler multiplier and, for $a = 1$ and $a = -1$, determine the phase portrait (be careful with the sign for $x > 0$ and $x < 0$). Locate periodic orbits, equilibria, and determine the omega-limit set for all initial conditions. Note it is alright to characterize some trajectories qualitatively and just draw their shapes. Also feel free to use any computational methods to explore the phase space.

Problem 2: Set $x' = Ax$, with $A = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$, $x \in \mathbb{R}^2$, and $a, b > 0$.

(a) Derive an equation for the "projectivized flow". That is write $x = \theta r$ with $r = |x|$ and vector $\theta \in \mathbb{R}^2$ with $|\theta| = 1$ and derive a differential equation for θ (here you could also view $\theta \in S^1$). More concretely, you could alternatively use polar coordinates $x = (r\cos\phi, r\sin\phi)^T$ and derive an equation for ϕ (note: why is ϕ' independent of r?)

(b) What are the equilibria of this flow? What are their eigenvalues and stability properties?

(c): Do the same for $A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$ for all $\mu \in \mathbb{R}$. Discuss how this approach helps qualitatively "unfold" or determine the dynamics for $\mu 0$ near 0 (where there is a two-dimensional Jordan block) .

Remark: This flow gives how one-dimensional linear subspaces evolve under the flow of the above differential equation. In other words, the flow of the linear equation $x' = Ax$ induces a flow on the space of linear subspaces, i.e. the Grassmanian (useful in many settings such as heteroclinic bifurcation theory and geometric singular perturbation theory).

Problem 3*(Perko 1.8.6)*: Find the Jordan canonical form of each of the following matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

Problem 4: Consider an ODE $u' = f(u)$ with flow Φ_t and suppose $u = 0$ is an asymptotically stable equilibrium. Define the basin of attachtion

$$B = \{u_0; \lim_{t\to\infty} \Phi_t(u_0) = 0\}.$$

(a) Show that B is open and invariant.

(b) Show that the basin boundary ∂B is invariant.

(bonus): Define $\delta := dist(0, \partial B) = \inf_{y\in\partial B} |y|$ and explain why δ measures robustness of the equilibrium with respect to noisy time-dependent perturbations. It might be useful to consider a simple ODE such as $x' = x - x^3$ and the basins of the equilibria $x = \pm 1$.

These sets can in general be be incredibly complicated (i.e. look up riddled basin boundaries).

Problem 5: (Global solutions for global Lipshitz): Consider $u' = f(u)$, $u \in \mathbb{R}^n$ with f globally Lipshitz, that is $|f(u) - f(v)| \leq L|u - v|$ for all u, v.

(a): Show that $C^0_\eta := \{u : [0, \infty) \to \mathbb{R}^n \,|\, u \text{ is continuous}, \|u\|_\eta < \infty\}$, with the exponentially weighted norm

$$\|u\|_\eta = \sup_{t \geq 0} e^{\eta t} |u(t)|,$$

is a Banach space.

(b): Define

$$(Tu)(t) = u_0 + \int_0^t f(u(s)) ds,$$

and show that $T : C^0_\eta \to C^0_\eta$ is well-defined for $\eta < 0$ (i.e. when you allow solutions to possibly grow weakly exponentially fast at $+\infty$.)

(c): Determine η so that T is a contraction and hence conclude global existence of solutions.