**MA775 - Fall 2023 Homework 4** - Due December 5th

**Problem 1***(Teschl 8.13)*: Consider $L(v, q) = \frac{1}{2}v^T M v - U(q)$ in $\mathbb{R}^3$ with $M = mI_3, m > 0$ and suppose the potential $U(q) = U(|q|)$ is rotation invariant. Show that the angular momentum $l = x \wedge p := x \times p$ is conserved in this case.

**Problem 2** *(Teschl 8.18)*: Consider the Hamiltonian $H(p, q) = \sum_{j=1}^{n} \frac{p_j^2}{2m} + U_0(q_{j+1} - q_j)$ with *interaction potential function* $U_0(x) = \frac{k}{2}(x^2 + \alpha x^3)$, and $q_0 = q_{n+1} = 0$, setting $m = k = 1$. Compute the eigenvalues and eigenvectors of the linear hamiltonian system with $\alpha = 0$. Set $N = 32, \alpha = 1/6$ and choose an initial condition in an eigenspace and numerically compute the time evolution. Investigate how the state is distributed with respect to the eigenvectors as a function of $t$ (here, you can measure this by tracking the quantities $c_j(t) = X(t) \cdot V_j$ of the solution vector $X(t) = (q(t), p(t))^T$ with each eigenvector $V_j$ as time evolves. Please reach out to me if you need some help solving this ODE numerically!

**Problem 3***(Guckenheimer & Holmes 4.3.2)*: Study the following systems using averaging methods:

$$x' = \epsilon(x - x^2)\sin^2 t, \qquad x' = \epsilon(x \sin^2 t - x^2/2).$$

**Problem 4***(Chicone Ex. 1.76)* Let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. A *Gradient System* takes the form

$$x' = -\nabla G(x).$$

Show that gradient systems can't have periodic orbits. Also show that if $x_* \in \mathbb{R}^n$ is an isolated minimum of $G$ then $x_*$ is an asymptotically stable equilibrium.