MA775 - Fall 2023 Midterm - Due October 26th

You must work on these problems on your own. You're welcome to use references but must cite whatever you use. Of course, please reach out if you have any questions.

Problem 1: Let

$$x' = \mu - x^2 + 2y^2, \quad y' = -y + x + \mu - x^2,$$

Study the parameter-dependent center manifold at the equilibrium $(x, y; \mu) = (0, 0, 0)$ for $\mu \sim 0$. In particular append an additional equation, $\mu' = 0$ to the system, and apply the center manifold theorem to obtain a second-order expansion of the center manifold. Calculate the leading order (i.e. the 2nd-order truncation) equation on the center manifold, and determine its qualitative dynamics, and therefore the dynamics in a neighborhood of $(x, y) = (0, 0)$ for $\mu \sim 0$.

Problem 2: Let

$$x' = -x + ay^2, \quad y' = -2y + bx^2.$$

(a) Draw the phase portrait of the linearization about the origin, and express all trajectories as graphs $x = h(y)$ or $y = h(x)$ (Hint: try dividing the two equations to obtain a differential equation(s) of the form $dy/dx =$ or $dx/dy = f(x, y)$).

(b) Find the Taylor expansion of the (smooth) strong stable manifold $x = h_{ss}(y)$ for the nonlinear system up to order 2.

(c) Try to find a one-dimensional "weak-stable" manifold tangent to $\{y = 0\}$, corresponding to the weak-stable eigenvalue $\lambda = -1$, by calculating a quadratic Taylor expansion. What goes wrong?

(d) Set $a = 0$ and compute solutions explicitly and show there exist manifolds $y = h_s(x)$, but none is C^2 because of terms of the form $x^2 logx$.

Problem 3 *(Non-analyticity of center manifold)*: Consider

$$x' = -x^3, \qquad y' = -y + x^2,$$

which has an analytic vector-field. Show that any center-manifold (tangent to the x-axis) is not analytic. To do this, set $h(x) = \sum_{j=2}^{\infty} c_j x^j$ and determine formula(s) for the coefficients c_j, what properties does this power series have?