MA775 - Fall 2023 Midterm 2 - Due December 12th (no late penalty for turn-in through Dec. 15th)

You must work on these problems on your own. You're welcome to use references but must cite whatever you use. Of course, please reach out if you have any questions.

Problem 1(Chicone 2.134/2.137) Consider the following ODE in polar coordinates.

$$r' = r(1-r), \quad \theta' = r.$$

Set $S = \{\theta = 0, r \geq 0\}$ to be the section and compute the associated Poincare map. Determine the eigenvalue of the linearization of this map at the fixed point $r = 1$. Also, for each p in the periodic orbit, compute the sets in the plane which have the same asymptotic phase as the solution starting at p.

Problem 2 Consider a system of the form

$$\begin{aligned} x' &= \lambda_1 x + o(|x| + |y|) \\ y' &= \lambda_2 y + o(|x| + |y|), \end{aligned}$$

with $\lambda_1 = 2\lambda_2$. Determine the normal form up to second order.

Problem 3*(Degenerate Pitchfork):* Determine the bifurcation diagram of the equation:

$$x' = \mu_1 x + \mu_2 x^3 - x^5.$$

That is, for all $(\mu_1, \mu_2) \in \mathbb{R}^2$, find all equilibria and determine their stability.

Problem 4: Consider the following system with parameter $\alpha \in \mathbb{R}$.

$$\begin{aligned} x' &= y, \\ y' &= x - x^2 + \alpha y^3 \end{aligned}$$

(i) Find and classify all equilibria for various $\alpha \in \mathbb{R}$

(ii) Find a change of coordinates which diagonalizes the linearized system about the origin.

(iii) Write the full nonlinear system in the coordinates from part (ii) and find a fourth-order approximation for the stable and unstable manifolds $W^{\mathrm{s}}(0), W^{\mathrm{u}}(0)$, respectively.

(iv) Using the calculations from part (c), another argument, or numerical investigation, find an approximate α value such that these manifolds intersect away from the origin. This gives an approximate value for the existence of a homoclinic orbit.

Problem 5(Compactification and asymptotics)[Credit to Tasso Kaper for this problem!]: Consider Airy's equation

$$\frac{d^2x}{dt^2} - tx = 0.$$

We wish to use a compactification/desingularization technique to understand the dynamics of solutions as $t \to +\infty$.

(a): Write the system as a first-order (non-autonomous) system, and introduce the variable $\beta = (t)^{-1/2}$. Derive an autonomous 3D-system of equations for $x, y := x'$, and β. The equilibrium $(x, y, \beta) = (0, 0, 0)$ represents the behavior near $(x, y) = (0, 0)$ at $t \to +\infty$ and represents a compactification of the point $t = \infty$.

(b) Desingularize by scaling $x = \beta^{-\alpha}\tilde{x}, y = \beta^{-\gamma}\tilde{y}, \frac{d}{dt} = \beta^{-\nu}\frac{d}{d\tau}$ so that the vector field is smooth in the new variables (that is find suitable α, γ, and ν so that the new vector field is smooth) and the leading order linear system when $\beta = 0$ for $(\tilde{x}, \tilde{y})$ is $\tilde{x}_\tau = \tilde{y}, \ \tilde{y}_\tau = \tilde{x}$.

(c) Linearize about the origin in the new 3D-system you found, find the expansion for the center-stable manifold up to order 4 and derive an equation for the dynamics on this manifold. Shortcut: you could write the center-stable manifold as a graph $\tilde{y} = h(\tilde{x}, \beta)$, with $h(0, 0) = 0$ and $\partial_{\tilde{x}} h(0, 0) = -1$ (tangency with stable subspace). Further hint: The center equation/dynamics should just be the β equation, and the dynamics of the other coordinate give the exponentially attracting behavior.

(d) The equation you get for the stable dynamics should take the leading order form

$$\tilde{x}_\tau = (-1 - c_1\beta^3)\tilde{x}$$

for appropriate constant c_1. Plug in the explicit form of $\beta(\tau)$ to this equation and solve for $\tilde{x}$. Unwind your scalings to derive an expansion for $x(t)$ as $t \to +\infty$. This should be the well-known asymptotics of the Airy function.

Do one of the following two problems:

Problem 6(a) *(Hopf persistence proof)[Iooss& Adelmayer]*:

As in class, consider $x' = F(z, \mu)$, $x \in \mathbb{R}^n, \mu \in \mathbb{R}, F(0, 0) = 0$ such that $L := D_x F(0, 0)$ has simple eigenvalues $\pm \mathrm{i}\omega, \omega > 0$ with eigenvectors $e_0, \overline{e_0}$ and with $\mathrm{spec}(L) \cap \mathrm{i}\mathbb{R} = \{\pm \mathrm{i}\omega\}$. As discussed in class, a center manifold reduction on the center subspace $E_0 = \{ze_0 + \overline{ze_0}\}_{z \in \mathbb{C}}$, along with a normal form reduction, gives the reduced system

$$\begin{aligned} z' &= \mathrm{i}\omega z + z g(|z|^2, \mu) + o(|z|^{2k+2}), \\ \bar{z}' &= -\mathrm{i}\omega \bar{z} + \bar{z}\overline{g(|z|^2, \mu)} + o(|z|^{2k+2}), \end{aligned}$$

with g a complex polynomial of order less or than equal to k in $|z|^2$, such that $g(0, 0) = 0$. Introducing polar coordinates $z = re^{\mathrm{i}\phi t}$ we find

$$\begin{aligned} r' &= r g_r(r^2, \mu) + R_r \\ \phi' &= \omega + g_\phi(r^2, \mu) + R_\phi \end{aligned}$$

with $g_r = (g + \bar{g})/2 := \sum_{j=0}^k a_j(\mu) r^{2j}, \quad g_\phi = (g - \bar{g})/2 := \sum_{j=0}^k b_j(\mu) r^{2j}$ and $R_r = o(r^{2k+2}), R_\phi = o(r^{2k+1})$.

(a): Show that the system, truncated at order $2k + 1$, undergoes a pitchfork bifurcation with direction determined by the leading order coefficients in g_r. You should find for $a := \frac{a_0'(0)}{a_1(0)} < 0$ the bifurcating solution takes the form, for $\mu > 0$,

$$r_*(\mu) = \sqrt{-a\mu} + o(|\mu|)), \quad \omega_*(\mu) = \omega + (b_0'(0) - b_1(0)a)\mu + o(|\mu|).$$

(b): Show that, for all (r, ϕ, μ) near zero, any solution must have $\frac{d\phi}{dt} > 0$. Use this to eliminate time, considering r as a function of ϕ, and derive a new ODE

$$\frac{dr}{d\phi} = \tilde{f}(r, \phi, \mu)$$

with $\tilde{f}$, 2π-periodic in ϕ.

(c): Let $r(0, \rho, \mu) = \rho$ be the initial data of solutions to this new ODE. Define a Poincare map, $(\rho, \mu) \mapsto P(\rho, \mu) := r(2\pi; \rho, \mu)$. Show that P can be factored $P(\rho, \mu) = \rho \tilde{P}(\rho, \mu)$ with smooth $\tilde{P}$ near $\rho, \mu \sim 0$ so that non-trivial fixed points of P are found by solving $\tilde{P}(\rho, \mu) = 1$.

(d): Show that $\tilde{P}(0, 0) = 1$ and use the implicit function theorem to solve $0 = \tilde{F}(\rho, \mu) := \tilde{P}(\rho, \mu) - 1$ for μ (in terms of ρ) near $(0, 0)$. Conclude the existence of a periodic orbit.

Hint: show that the function $\phi \mapsto \frac{\partial r}{\partial \rho}(\phi; 0, \mu)$ satisfies the ODE $\partial_\phi(\frac{\partial r}{\partial \rho}(\phi; 0, \mu)) = \frac{a_0(\mu)}{\omega + b_0(\mu)} \frac{\partial r}{\partial \rho}(\phi; 0, \mu)$ and conclude that $\partial_\rho r(\phi; 0, \mu) = \mathrm{e}^{(\frac{a_0(\mu)}{\omega + b_0(\mu)})\phi}$.

Problem 6(b) *(Bogdanov-Takens Normal form/The double-zero eigenvalue)*:

Consider the following system near $\underline{x} := (x, y) = (0, 0)$, with small parameters $\mu \sim 0$:

$$\begin{aligned} \dot{x} &= y + \mathcal{O}(|\mu| + |\underline{x}|^2) \\ \dot{y} &= 0 + \mathcal{O}(|\mu| + |\underline{x}|^2). \quad , \end{aligned}$$

(a): First study the normal form, up to second order, for $\mu = 0$, showing the above system can be transformed into the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^2 + xy + \mathcal{O}(|\underline{x}|^3). \quad , \end{aligned} \tag{0.1}$$

e:sys1

(Hint: One can proceed systematically as in [Guckenheimer & Holmes §3.3], studying the range of the homological operator.)

(b): (Unfolding) It can be found that the parameter dependent unfolding

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y - x^2 + xy + \mathcal{O}(|\underline{x}|^3), \quad \mu_1, \mu_2 \in \mathbb{R}, \end{aligned}$$

is a *versal deformation* of the vector-field. That is there exists a C^0-conjugacy between the linear part of this system and any small perturbation of the system (0.1) in a neighborhood of the origin (See [Arnold, Geometric Methods...] for more detail). Omitting the $\mathcal{O}(3)$-terms, study the stability properties of the equilibria $(x, y) = (\pm\sqrt{\mu_1}, 0)$, in the parameter plane $(\mu_1, \mu_2) \in \mathbb{R}^2$ and locate any curves in this plane where bifurcations occur. As much as you can, sketch the phase-portraits of the vector fields in each of the main parameter regions.

(c): There exists another bifurcation curve corresponding to a global bifurcation where the limit cycle vanishes along a homoclinic orbit. To study this, we wish to scale around this area: $\mu_1 \sim \mu_2^2$. That is scale as following: $x = \tilde{x}\epsilon^2, \quad y = \tilde{y}\epsilon^3, \quad \mu_2 = -\epsilon^2\nu_2, \mu_1 = \nu\epsilon^4, \frac{d}{d\tau} = \frac{d}{dt}\epsilon$, so that

$$\begin{aligned} \tilde{x}' &= y, \\ \tilde{y}' &= \nu - \epsilon\nu_2\tilde{y} + \epsilon\tilde{x}\tilde{y} - \tilde{x}^2. \end{aligned} \tag{0.2}$$

For $\epsilon = 0$ one obtains the "anharmonic" oscillator (dropping tilde's)

$$
\begin{aligned}
x' &= y, \\
y' &= \nu - x^2,
\end{aligned}
\tag{0.3}
$$

Find the Hamiltonian for this system. For $\nu = 1$, the system has a homoclinic orbit explicitly given by

$$
(x(t), y(t)) = (3\mathrm{sech}^2(t/\sqrt{2}) - 1, -3\sqrt{2}\mathrm{sech}^2(t/\sqrt{2})\mathrm{tanh}(t/\sqrt{2})).
$$

Use the method of Melnikov to approximately find the parameter curve for which the homoclinic persists under the perturbation $\epsilon(0, -\nu_2\tilde{y} + \tilde{x}\tilde{y})$. Hint: One should approximately find $\nu_2 = 7/5$ and thus $\mu_1 = (5\mu_2/7)^2$ for $\mu_2 < 0$.

See [Guckenheimer & Holmes], [Kuznetsov], [Takens 1974], or [Bogdanov 1975] for more detail.