MA776 - Spring 2022 Homework 2 - Due Friday, Feb. 18th

Please scan or type your homework and email it to me by the due date. Homeworks will be graded electronically and emailed back to you.

Evans Problems

Chapter 2: 5 (see appendices for definition of convexity), 6, 9, 17

Additional Problems :

Problem 1:

Consider Poisson's Kernel, $K(x,y) = \frac{2x_n}{n\alpha(n)}|x-y|^{-n}$ for $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$. Show that $\int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1$.

Problem 2: (Comparison principle):

Consider the heat equation $u_t = \Delta u$, for $x \in U$ an open, and bounded set in $\mathbb{R}^n$ with C^1 boundary. Let u and v be two solutions, both C^2 in x and C^1 in t, and assume that $u(x,t) \geq v(x,t)$, for all $(x,t) \in \Gamma_T$ for some positive time $T > 0$. Then prove that $u(t,x) \geq v(t,x)$ for all $(x,t) \in U_T$.

Problem 3: (Moments):

Consider the heat equation $u_t = \Delta u$ with domain $U = \mathbb{R}^n$ and initial data $u(x,t) = g(x)$ with $g \in C^\infty(\mathbb{R}^n)$ and both $g(x), |x|^2 g(x) \in L^1(\mathbb{R}^n)$.

(a): Show that the "mass" $M_0(t) := \int_{R^n} u(x,t)dx$ is conserved for all time $t \geq 0$.

(b): Now consider the "second moment" $M_2(t) = \int_{R^n} |x|^2 u(x,t)dx$. Calculate an explicit form of $M_2(t)$ interms of $M_2(0)$ and M_0.

(c) (optional): Use the expression from (b) to try to describe how the distribution of mass of the solution (i.e. the regions in $\mathbb{R}^n$ where $u(x,t)$ is most non-zero) evolves over time. For example, try to get a bound for the mass outside the ball $B(0,r)$ with $r > Ct^{1/2}$, for $C > 0$ large using your expression for $M_2(t)$.