Some examples of the Riemann-Hilbert correspondence

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1 Introduction

1.1 Fix a variety X over $\mathbb{C}$. The Riemann-Hilbert correspondence identifies the category of perverse sheaves on $X(\mathbb{C})$ with the (abelian) category of regular holonomic $\mathscr{D}$-modules on X. This is a remarkable and deep theorem in the theory of linear partial differential equations. In this note we will investigate this correspondence in simple examples, exploring the topological and algebraic sides as well as their connection through analysis.

1.2 Let us define the abelian category of $\mathscr{D}$-modules on a smooth variety X. First, the *sheaf of differential operators* $\mathscr{D}_X$ is the quasicoherent sheaf of associative algebras defined inductively as follows. The differential operators of order 0 make up the structure sheaf $\mathscr{O}_X$. A local section $D \in \mathscr{E}nd_{\mathbb{C}}(\mathscr{O}_X)$ is a differential operator of order ≤ 1 provided that $[D, f] \in \mathscr{O}_X \subset \mathscr{E}nd_{\mathbb{C}}(\mathscr{O}_X)$ for any $f \in \mathscr{O}_X$. We say that D is a differential operator of order $\leq n$ if $[D, f]$ is a differential operator of order $\leq n-1$ for any $f \in \mathscr{O}_X$, and $\mathscr{D}_X$ consists of differential operators of all orders. In particular $\mathscr{D}_X$ has a canonical filtration, and the associated graded sheaf of algebras is $\operatorname{Sym}_{\mathscr{O}_X} \mathscr{T}_X$.

A sheaf of left $\mathscr{D}_X$-modules $\mathscr{M}$ is called a *$\mathscr{D}$-module on X* if $\mathscr{M}$ is quasicoherent as an $\mathscr{O}_X$-module. A $\mathscr{D}$-module is called *coherent* if it is locally finitely generated as an $\mathscr{D}_X$-module. A coherent $\mathscr{D}$-module $\mathscr{M}$ is known to admit a filtration compatible with the filtration on $\mathscr{D}_X$, so one obtains a module over $\operatorname{Sym}_{\mathscr{O}_X} \mathscr{T}_X$. The support of this module does not depend on the chosen filtration and is called the *characteristic variety* or *singular support* $\operatorname{Sing}(\mathscr{M})$: it is a closed subvariety of the cotangent bundle T^*X which is conical, i.e. stable under the $\mathbb{G}_m$-action. One can prove that $\dim \operatorname{Sing}(\mathscr{M}) \geq \dim X$, and we call $\mathscr{M}$ *holonomic* provided that $\dim \operatorname{Sing}(\mathscr{M}) = \dim X$, i.e. the characteristic variety has minimal possible dimension.

In order to state the Riemann-Hilbert correspondence one has to define what it means for a holonomic $\mathscr{D}$-module to have regular singularities. Roughly speaking, this means that the solutions of a corresponding system of differential equations have moderate (i.e. at most polynomial) growth. Rather than give a precise definition, we give an example of a $\mathscr{D}$-module which does not have regular singularities. The global differential operators on $\mathbb{A}^1$ are identified with the Weyl algebra W_1, which is the free noncommutative algebra on the symbols x and ∂ modulo the relation $x\partial - \partial x = 1$. The *exponential $\mathscr{D}$-module* exp, defined by $\Gamma(\mathbb{A}^1, \exp) = W_1/W_1(\partial - 1)$, does not have regular singularities because its solutions do not have moderate growth at infinity.

1.3 Before proceeding further, we will state the theorem in greater generality. On one side we consider the full subcategory $\mathscr{D}^{\mathrm{rs}}_{\mathrm{hol}}(X)$ of the bounded derived category of $\mathscr{D}_X$-modules on X consisting of complexes whose cohomologies are holonomic with regular singularities. On the other is the full subcategory $D^b_c(X)$ of the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on $X(\mathbb{C})$ consisting of complexes whose cohomologies are constructible with respect to some algebraic stratification of X.

For simplicity, let us assume that X is smooth. Given a complex of $\mathscr{D}$-modules $\mathscr{M} \in \mathscr{D}(X)$, one defines the *solution complex* as

$$\operatorname{Sol}(\mathscr{M}) := R\mathscr{H}om_{\mathscr{D}_{X^{\mathrm{an}}}}(\mathscr{M}^{\mathrm{an}}, \mathscr{O}_{X^{an}}).$$

A theorem of Kashiwara ensures that if $\mathscr{M}$ is cohomologically bounded with holonomic cohomology sheaves, then $\operatorname{Sol}(\mathscr{M}) \in D^b_c(X)$. It is so named because if X is affine and $\mathscr{I} \subset \mathscr{D}_X$ is a left ideal, generated by some differential operators $D_1, \cdots, D_n$, then $H^0 \operatorname{Sol}(\mathscr{D}_X)$ is the sheaf of holomorphic functions $f \in \mathscr{O}_{X^{\mathrm{an}}}$ satisfying the differential equations

$$D_1 f = \cdots = D_n f = 0.$$

Theorem 1.3.1 (Riemann-Hilbert correspondence). *The functor* Sol *restricts to an equivalence*

$$\mathscr{D}^{\mathrm{rs}}_{\mathrm{hol}}(X)^{\mathrm{op}} \xrightarrow{\sim} D^b_c(X).$$

To obtain a covariant equivalence one can compose with Verdier duality, and the resulting functor agrees with the analytic de Rham complex, which is therefore an equivalence also.

1.4 Both sides of the equivalence in Theorem 1.3.1 are equipped with natural t-structures, since they are categories of complexes. However, the functor is not compatible with these t-structures, not even up to shift. To get an equivalence of categories with t-structures, and hence an equivalence between their abelian hearts, one uses the *perverse t-structure* on $D^b_c(X)$. Rather than give a definition, we remark that this t-structure is determined by the equivalence Sol and the natural t-structure on $\mathscr{D}^{\mathrm{rs}}_{\mathrm{hol}}(X)$, and in particular the abelian category of perverse sheaves can be defined as the essential image of the abelian category of regular holonomic $\mathscr{D}$-modules under the functor Sol (or better, $\operatorname{Sol}[\dim X]$).

2 Local systems on the punctured affine line

2.1 Recall that a *local system* is a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces. Perverse sheaves, at least on a smooth variety, are supposed to be generalized local systems which are allowed to be "singular" along some stratification. In order to understand general perverse sheaves we should first study *lisse* perverse sheaves, which come from local systems. We will do this now in the case $X = \mathbb{A}^1 \setminus \{0\}$, so $X(\mathbb{C}) = \mathbb{C}^\times$.

A local system on $\mathbb{C}^\times$ corresponds to a finite-dimensional representation of the fundamental group $\pi_1(\mathbb{C}^\times, 1) = \mathbb{Z}$, i.e. a finite-dimensional vector space V equipped with a linear automorphism φ. This automorphism is called the *monodromy* of the local system. The locally constant sheaf corresponding to (V, φ) is the sheaf of sections of the space $(\mathbb{C} \times V)/\mathbb{Z}$, where $\mathbb{Z}$ acts diagonally, by translations on $\mathbb{C}$ and by φ on V. In other words, if we view the universal cover $\mathbb{C} \to \mathbb{C}^\times$ given by $x \mapsto e^{2\pi i x}$ as a $\mathbb{Z}$-bundle, the desired local system is the sheaf with fiber V associated with the $\mathbb{Z}$-representation (V, φ). Isomorphism classes of pairs (V, φ) are classified by Jordan normal forms modulo permutation of the Jordan blocks. In particular, the simple objects $\mathscr{L}_\mu$ are parameterized by $\mu \in \mathbb{C}^\times$.

2.2 Which $\mathscr{D}$-module corresponds to the lisse perverse sheaf $\mathscr{L}_\mu$? The global differential operators on $\mathbb{A}^1 \setminus \{0\}$ are $W_1[x^{-1}]$ (see Section 1 for the definition of the Weyl algebra W_1). Because $\mathbb{A}^1 \setminus \{0\}$ is affine, $\mathscr{D}$-modules are just $W_1[x^{-1}]$-modules, and for any $\lambda \in \mathbb{C}$ we have the $\mathscr{D}$-module

$$x^\lambda := W_1[x^{-1}]/W_1[x^{-1}](x\partial - \lambda).$$

Notice that the $\mathbb{C}[x, x^{-1}]$-module underlying x^λ is free of rank one, i.e. the trivial line bundle, and the $\mathscr{D}$-module structure is the flat connection determined by

$$\nabla \cdot 1 = \frac{\lambda}{x}.$$

Under the Riemann-Hilbert correspondence x^λ is sent to $\mathscr{L}_\mu$, where $\mu = e^{2\pi i \lambda}$.

To see why, choose a connected and simply connected open set $U \subset \mathbb{C}^\times$ and a branch $\log : U \to \mathbb{C}$ of the complex logarithm. One checks that all solutions on U of the differential equation

$$f'(x) = \frac{\lambda}{x} f(x)$$

are scalar multiples of $f(x) = e^{\lambda \log x}$, and in particular the $\mathscr{D}$-module x^λ corresponds to a local system of rank 1. To compute the monodromy, fix $p \in U$ and a loop γ based at p which winds once around the origin counterclockwise. Parallel transport along γ adds $2\pi i$ to log, so $f(x)$ is sent to

$$e^{\lambda(2\pi i + \log x)} = e^{2\pi i \lambda} f(x).$$

3 Perverse sheaves on the projective line

3.1 Now we study the situation on a curve where we allow sheaves to be singular at a single point. We may as well take $X = \mathbb{P}^1$, so that $X(\mathbb{C})$ is the Riemann sphere, and allow singularities only at ∞. That is, we consider the category $\mathrm{Perv}_\infty(\mathbb{P}^1)$ of perverse sheaves on $\mathbb{P}^1$ whose restriction to $\mathbb{A}^1$ is lisse (here and elsewhere we abuse notation and ignore the distinction between $X(\mathbb{C})$ and X). Since $\mathbb{A}^1$ is simply connected, local systems and hence lisse perverse sheaves on $\mathbb{A}^1$ are all trivial. Thus $\mathrm{Perv}_\infty(\mathbb{P}^1)$ is determined by the local geometry around ∞. As with any category of perverse sheaves $\mathrm{Perv}_\infty(\mathbb{P}^1)$ is Artinian, meaning every object admits a finite filtration with simple subquotients. We will see that there are two simple objects and five indecomposable objects.

3.2 The two simple objects in $\mathrm{Perv}_\infty(\mathbb{P}^1)$ are the IC (intersection cohomology) sheaf $\mathrm{IC}_{\mathbb{P}^1}$, which comes from the trivial local system, and the delta sheaf δ_∞. Letting $j : \mathbb{A}^1 \to \mathbb{P}^1$ be the inclusion, we have pushforward functors j_* and $j_!$, and in particular we can consider $j_* \mathrm{IC}_{\mathbb{A}^1}$ and $j_! \mathrm{IC}_{\mathbb{A}^1}$. These sheaves are indecomposable, and fit into short exact sequences

$$0 \longrightarrow \mathrm{IC}_{\mathbb{P}^1} \longrightarrow j_* \mathrm{IC}_{\mathbb{A}^1} \longrightarrow \delta_\infty \longrightarrow 0$$

and

$$0 \longrightarrow \delta_\infty \longrightarrow j_! \mathrm{IC}_{\mathbb{A}^1} \longrightarrow \mathrm{IC}_{\mathbb{P}^1} \longrightarrow 0.$$

There is still another indecomposable object $\mathscr{P}_\infty$ in $\mathrm{Perv}_\infty(\mathbb{P}^1)$, which is not immediately accessible via the usual six functors formalism. The sheaf $\mathscr{P}_\infty$ is sometimes called the *big projective* because it is projective and it is the longest indecomposable object. The easiest way to characterize the big projective is as follows: a simple object of an Artinian category with enough projectives receives an epimorphism from an indecomposable projective object, unique up to non-unique isomorphism, called its *projective cover*. The projective cover of δ_∞ is $\mathscr{P}_\infty$. Dually, in an Artinian category with enough injectives simple objects have *injective hulls*, and $\mathscr{P}_\infty$ is also the injective hull of δ_∞. Thus $\mathscr{P}_\infty$ contains δ_∞ as both a subobject and a quotient. In fact there is a filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 = \mathscr{P}_\infty$$

where $F_1 \cong \delta_\infty$, $F_2/F_1 \cong \mathrm{IC}_{\mathbb{P}^1}$, and $\mathscr{P}_\infty/F_2 \cong \delta_\infty$.

There are many more characterizations of $\mathscr{P}_\infty$, which is easily the most interesting object of $\mathrm{Perv}_\infty(\mathbb{P}^1)$. Here is one more. Letting $i : \{\infty\} \to \mathbb{P}^1$ be the inclusion, there are two pullback functors $i^!$ and i^*, which take values in complexes of vector spaces. For example, $i^! \mathrm{IC}_{\mathbb{P}^1} = \mathbb{C}[-1]$ and $i^* \mathrm{IC}_{\mathbb{P}^1} = \mathbb{C}[1]$. An object $\mathscr{F}$ of $\mathrm{Perv}_\infty(\mathbb{P}^1)$ is called *tilting* if both $i^!\mathscr{F}$ and $i^*\mathscr{F}$ have cohomology concentrated in degree zero. Thus δ_∞ is tilting because $i^!\delta_\infty = \mathbb{C} = i^*\delta_\infty$, whereas $\mathrm{IC}_{\mathbb{P}^1}$ is not. One checks that $i^!\mathscr{P}_\infty = \mathbb{C} = i^*\mathscr{P}_\infty$, so the big projective is tilting. In fact, $\mathscr{P}_\infty$ is the unique indecomposable tilting sheaf which restricts to $\mathrm{IC}_{\mathbb{A}^1}$. This follows from the calculations $i^* j_* \mathrm{IC}_{\mathbb{A}^1} = \mathbb{C} \oplus \mathbb{C}[1]$ and $i^! j_! \mathrm{IC}_{\mathbb{A}^1} = \mathbb{C} \oplus \mathbb{C}[-1]$ (the base change formula implies that $i^! j_* = 0 = i^* j_!$).

3.3 Having described the indecomposable objects of $\mathrm{Perv}_\infty(\mathbb{P}^1)$ and computed the relevant local invariants (namely !- and $*$-fibers at ∞), we now calculate global invariants of these sheaves. More precisely, writing $p : \mathbb{P}^1 \to \mathrm{pt}$, there is a pushforward functor $p_* = p_!$ which takes values in complexes of vector spaces. Unsurprisingly $p_*\delta_\infty = p_* i_* \mathbb{C} = \mathbb{C}$. We have $p_* j_* \mathrm{IC}_{\mathbb{A}^1} = \mathbb{C}[1]$ because, up to the shift that appears because $\mathrm{IC}_{\mathbb{A}^1}[-1]$ is the constant sheaf, this complex computes the cohomology of $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$. Similarly $p_* \mathrm{IC}_{\mathbb{P}^1} = \mathbb{C}[1] \oplus \mathbb{C}[-1]$ computes the cohomology of $\mathbb{P}^1(\mathbb{C})$ shifted by 1. The complex $p_* j_! \mathrm{IC}_{\mathbb{A}^1} = \mathbb{C}[-1]$ computes the homology of $\mathbb{A}^1$, up to the shift that appears because $\mathrm{IC}_{\mathbb{A}^1}[1]$ is the dualizing sheaf.

We expect a more interesting answer for $\mathscr{P}_\infty$, and in fact $p_*\mathscr{P}_\infty = 0$. In order to prove this we need Verdier duality, which is an endofunctor $\mathbb{D}$ of $\mathrm{Perv}_\infty(\mathbb{P}^1)$ with the property that $\mathbb{D} \circ \mathbb{D}$ is canonically isomorphic to the identity functor. For example, $\mathbb{D}\delta_\infty = \delta_\infty$ and $\mathbb{D}\,\mathrm{IC}_{\mathbb{P}^1} = \mathrm{IC}_{\mathbb{P}^1}$. Since $\mathbb{D} j_* = j_! \mathbb{D}$, we have $\mathbb{D} j_* \mathrm{IC}_{\mathbb{A}^1} = j_! \mathrm{IC}_{\mathbb{A}^1}$ and $\mathbb{D} j_! \mathrm{IC}_{\mathbb{A}^1} = j_* \mathrm{IC}_{\mathbb{A}^1}$. Now because $\mathbb{D}$ is an equivalence $\mathbb{D}\mathscr{P}_\infty$ must be indecomposable of length three, which means there is an isomorphism $\mathbb{D}\mathscr{P}_\infty \cong \mathscr{P}_\infty$. We have $p_*\mathbb{D} = \mathbb{D} p_*$ because p is proper, so $p_* \mathscr{P}_\infty$ is self-dual as well (in the derived category of vector spaces). The aforementioned filtration on $\mathscr{P}_\infty$ yields an exact sequence

$$0 \longrightarrow \delta_\infty \longrightarrow \mathscr{P}_\infty \longrightarrow j_* \operatorname{IC}_{\mathbb{A}^1} \longrightarrow 0,$$

and after applying p_* we obtain an exact triangle

$$\mathbb{C} \longrightarrow p_* \mathscr{P}_\infty \longrightarrow \mathbb{C}[1].$$

Thus either $p_* \mathscr{P}_\infty = 0$ or $p_* \mathscr{P}_\infty \cong \mathbb{C} \oplus \mathbb{C}[1]$, and self-duality rules out the latter possibility.

4 $\mathscr{D}$-modules on the projective line

4.1 Now we discuss regular holonomic $\mathscr{D}$-modules on $\mathbb{P}^1$ which are lisse away from ∞, this category being equivalent to $\operatorname{Perv}_\infty(\mathbb{P}^1)$ under the Riemann-Hilbert correspondence. The following remarkable fact allows us to work with these objects very explicitly.

Proposition 4.1.1. *The functor of global sections factors through a t-exact equivalence*

$$\mathscr{D}(\mathbb{P}^1) \longrightarrow \Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})\text{-mod},$$

which therefore induces an equivalence of the hearts on each side.

One says that $\mathbb{P}^1$ is *$\mathscr{D}$-affine*, which is equivalent to the assertion that $\mathscr{D}_{\mathbb{P}^1}$ is a projective generator of the abelian category $\mathscr{D}(\mathbb{P}^1)^\heartsuit$. This may be surprising, because the corresponding claim for coherent sheaves on any projective variety (of nonzero dimension) is very far from true. Nonetheless, any partial flag variety, and in particular any projective space, is $\mathscr{D}$-affine.

4.2 We would therefore like an explicit description of the associative algebra $\Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})$. As preparation, let us first consider the Lie algebra $\Gamma(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1})$. The standard Euler sequence shows that $\mathscr{T}_{\mathbb{P}^1}$ is canonically isomorphic to $\mathscr{O}_{\mathbb{P}^1}(2)$, whose global sections are three-dimensional, but we also want to compute the Lie bracket. The vector fields $x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}$, and $y\frac{\partial}{\partial y}$ (we omit the pushforward from $\mathbb{A}^2 \setminus \{0\}$ from the notation) span $\Gamma(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1})$, subject to the relation $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 0$. The bracket is determined by the usual rules of differentiation, but this Lie algebra has an even simpler and more familiar realization.

Recall that the Lie algebra $\mathfrak{sl}_2$ has a standard basis given by

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Proposition 4.2.1. *The action of* SL_2 *on* $\mathbb{P}^1$ *induces an isomorphism*

$$\mathfrak{sl}_2 \xrightarrow{\sim} \Gamma(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1}),$$

which sends

$$h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \; e \mapsto x\frac{\partial}{\partial y}, \; \textit{and } f \mapsto y\frac{\partial}{\partial x}.$$

Thus the Lie algebra morphism $\mathfrak{sl}_2 = \Gamma(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1}) \to \Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})$ induces a functor

$$\mathscr{D}(\mathbb{P}^1) \xrightarrow{\sim} \Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})\text{-mod} \longrightarrow \operatorname{Rep}(\mathfrak{sl}_2).$$

In fact, the image of $\mathfrak{sl}_2$ generates $\Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})$ as an associative algebra, so this functor is fully faithful. Thus we can view $\mathscr{D}$-modules on $\mathbb{P}^1$ as certain, generally infinite-dimensional, representations of $\mathfrak{sl}_2$.

Now we describe the essential image of this functor, i.e. which representations of $\mathfrak{sl}_2$ arise from $\mathscr{D}$-modules on $\mathbb{P}^1$. Recall that the center of the enveloping algebra $U(\mathfrak{sl}_2)$ is generated by the *Casimir element* $h^2 + 2ef + 2fe$. It is not hard to check by direct calculation that the morphism $U(\mathfrak{sl}_2) \to \Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})$ induced by the isomorphism of Proposition 4.2.1 sends the Casimir element to zero, and in fact we have the following.

Theorem 4.2.2. *The morphism $U(\mathfrak{sl}_2) \to \Gamma(\mathbb{P}^1, \mathscr{D}_{\mathbb{P}^1})$ is surjective with kernel generated by the Casimir element, and in particular induces an isomorphism*

$$U(\mathfrak{sl}_2)_0 := U(\mathfrak{sl}_2)/U(\mathfrak{sl}_2)(h^2 + 2ef + 2fe).$$

Thus the functor

$$\mathscr{D}(\mathbb{P}^1) \longrightarrow \mathrm{Rep}(\mathfrak{sl}_2)$$

is an equivalence onto the full subcategory of representations of $\mathfrak{sl}_2$ with trivial infinitesimal character, meaning $h^2 + 2ef + 2fe$ acts by zero.

4.3 Now we turn to the task of identifying the representations of $\mathfrak{sl}_2$ which correspond to the five indecomposable perverse sheaves from Section 3. The $\mathscr{D}$-module corresponding to $\mathrm{IC}_{\mathbb{P}^1}$ is just $\mathscr{O}_{\mathbb{P}^1}$ with the tautological action of $\mathscr{D}_{\mathbb{P}^1}$, and $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = \mathbb{C}$ with the trivial $\mathfrak{sl}_2$-action.

The next easiest representation to construct is actually the one corresponding to $j_! \,\mathrm{IC}_{\mathbb{A}^1}$. Let $\mathfrak{b} \subset \mathfrak{sl}_2$ be the Lie subalgebra of upper triangular matrices, so we can form the *Verma module*

$$M_0 := U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_0,$$

where $\mathbb{C}_0$ has the trivial action of $\mathfrak{b}$. The *highest weight vector* in M_0 is $v_0 := 1 \otimes 1$. For $n \geq 0$ put $v_{-2n} := f^n \cdot v_0$. Then $v_0, v_{-2}, \cdots$ is an eigenbasis for M_0 with respect to h, and $h \cdot v_{-2n} = -2nv_{-2n}$. Thus v_0 is called the highest weight vector because it has the greatest eigenvalue among eigenvectors of h. Obviously $f \cdot v_{-2n} = v_{-2n-2}$, and

$$e \cdot v_{-2n} = [e, f^n] \cdot v_0 = (nf^{n-1}h - n(n-1)f^{n-1})v_0 = -n(n-1)v_{-2n+2}.$$

Notice that M_0 is not irreducible: the vectors v_{-2n} for $n \geq 1$ span a subrepresentation generated by its own highest weight vector v_{-2}. In fact, this subrepresentation is isomorphic to the Verma module

$$M_{-2} := U(\mathfrak{sl}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-2},$$

where $\mathbb{C}_{-2}$ has the $\mathfrak{b}$-action determined by $e \cdot 1 = 0$ and $h \cdot 1 = -2$. We have a (nonsplit) exact sequence

$$0 \longrightarrow M_{-2} \longrightarrow M_0 \longrightarrow \mathbb{C} \longrightarrow 0,$$

so the irreducible representation M_{-2} corresponds to δ_∞.

The sheaf $j_* \,\mathrm{IC}_{\mathbb{A}^1}$ corresponds to the *dual Verma module* $M_0^\vee$, which has an eigenbasis $v_0^\vee, v_{-2}^\vee, \cdots$ with respect to h satisfying $h \cdot v_{-2n}^\vee = -2nv_{-2n}^\vee$, $e \cdot v_{-2n}^\vee = -v_{-2n+2}^\vee$ (where $v_{-2n} = 0$ for $n < 0$), and $f \cdot v_{-2n}^\vee = n(n+1)v_{-2n-2}^\vee$. As expected, it fits into an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow M_0^\vee \longrightarrow M_{-2} \longrightarrow 0.$$

Finally, we give an explicit description of the representation P_{-2} corresponding to $\mathscr{P}_\infty$. The vector space P_{-2} has a basis $v_0, v_{-2}, \cdots, w_{-2}, w_{-4}, \cdots$ consisting of eigenvectors for h, where $h \cdot v_{-2n} = -2nv_{-2n}$ and $h \cdot w_{-2n} = -2nw_{-2n}$. The rest of the action is given by $e \cdot v_{-2n} = -n(n-1)v_{-2n+2}$, $e \cdot w_{-2n} = -w_{-2n+2}$, $f \cdot v_{-2n} = v_{-2n-2}$, and $f \cdot w_{-2n} = n(n+1)w_{2n-2}$. There is an evident nonsplit exact sequence

$$0 \longrightarrow M_0 \longrightarrow P_{-2} \longrightarrow M_{-2} \longrightarrow 0.$$