# CLASSICAL MOTIVATION FOR THE RIEMANN–HILBERT CORRESPONDENCE

BRIAN CONRAD

These notes explain the equivalence between certain topological and coherent data on complex-analytic manifolds, and also discuss the phenomenon of "regular singularities" connections in the 1-dimensional case. It is assumed that the reader is familiar with the equivalence of categories between the category of locally constant sheaves of sets on a topological space $X$ and the category of covering spaces over $X$, and also (for a pointed space $(X, x_0)$) with the resulting equivalence of categories between the category of locally constant sheaves of sets on $X$ and left $\pi_1(X, x_0)$-sets when $X$ is connected and "nice" in the sense that it has a base of opens that are path-connected and simply connected. (For example, it is a hard theorem that any complex-analytic space is "nice"). These equivalences will only be used in the case of complex manifolds, but for purposes of conceptual clarity we give some initial construction without smoothness restrictions.

Let us first set some conventions. As usual in mathematics, we fix an algebraic closure $\mathbf{C}$ of $\mathbf{R}$, endowed with its unique absolute value extending that on $\mathbf{R}$. We shall work with arbitrary complex-analytic spaces (the analytic counterpart to $\mathbf{C}$-schemes that are locally of finite type), but the reader who is unfamiliar with this theory will not lose anything (nor run into problems in the proofs) by requiring all analytic spaces to be manifolds (in which case what we are calling a "smooth map" is a submersion by another name). For technical purposes it is important that the theory of $D$-modules works without smoothness restrictions. Hence, for the analytically-inclined reader who wishes to avoid smoothness restrictions we note that the global method of construction of the relative de Rham complex $\Omega^\bullet_{X/S}$ for any morphism of schemes $f : X \to S$ as in [5, $\mathrm{IV}_4$, §16.6] works *verbatim* in the complex-analytic case and is compatible with the analytification functor from locally finite type $\mathbf{C}$-schemes to complex-analytic spaces.

Notation and terminology. If $(X, \mathscr{O}_X)$ is a locally ringed space and $\mathscr{F}$ is an $\mathscr{O}_X$-module then $\mathscr{F}^\vee$ denotes $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ and $\mathscr{F}(x)$ denotes $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$; we call $\mathscr{F}(x)$ the *fiber* at $x$ (to be distinguished from the *stalk* $\mathscr{F}_x$ at $x$). We will freely pass between the categories of vector bundles and locally free sheaves, and so if $V$ is a vector bundle then we may write $V(x)$ to denote its fiber over a point $x$ in the base space. If $\Sigma$ is a set and $X$ is a topological space then $\underline{\Sigma}$ denotes the associated constant sheaf of sets on $X$.

The analytic space $\mathrm{Sp}\,\mathbf{C}$ is $\mathrm{Spec}\,\mathbf{C}$ considered as a 0-dimensional complex manifold. Also, $\mathbf{Z}(1)$ denotes the kernel of $\exp : \mathbf{C} \to \mathbf{C}^\times$; this is a free $\mathbf{Z}$-module of rank 1 but it does not have a canonical basis. Finally, a *local system* (of sets, abelian groups, *etc.*) on a topological space is a locally constant sheaf (of sets, abelian groups, *etc.*) on the space.

## 1. Basic definitions and constructions

Let $\pi : X \to Y$ be a map of analytic spaces, and $\mathscr{E}$ a coherent sheaf on $X$. A *connection* on $\mathscr{E}$ relative to $Y$ is a map of abelian sheaves

$$\nabla : \mathscr{E} \to \Omega^1_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{E}$$

that satisfies the Leibnitz rule

$$\nabla(fs) = \mathrm{d}_{X/Y} f \otimes s + f \cdot \nabla(s)$$

for local sections $s$ of $\mathscr{E}$ and $f$ of $\mathscr{O}_X$; in particular, $\nabla$ is $\mathscr{O}_Y$-linear. Such pairs $(\mathscr{E}, \nabla)$ form a category in an evident manner. The set of connections on $\mathscr{E}$ relative to $Y$ is a principal homogeneous space under $\mathrm{Hom}_X(\mathscr{E}, \Omega^1_{X/Y} \otimes \mathscr{E})$ because $\nabla - \nabla'$ is $\mathscr{O}_X$-linear for any two connections on $\mathscr{E}$ relative to $Y$.

**Example 1.1.** A basic example with $Y = \mathrm{Sp}\,\mathbf{C}$ is $\mathscr{E} = \mathscr{O}_X \otimes_{\underline{\mathbf{C}}} \Lambda$ for a local system $\Lambda$ of finite-dimensional $\underline{\mathbf{C}}$-vector spaces, taking $\nabla = \mathrm{d}_X \otimes 1$. This connection clearly kills $\Lambda$, and it is the unique connection on $\mathscr{E}$ with this property (due to the Leibnitz rule). If $X$ is a manifold then $\ker \mathrm{d}_X = \underline{\mathbf{C}}$, and so $\ker \nabla = \Lambda$ in the smooth case.

More generally, for any $Y$ we may consider $\mathscr{E} = \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} \Lambda$ for a locally free $\pi^{-1}\mathscr{O}_Y$-module $\Lambda$ with finite rank. In this case, $\nabla = \mathrm{d}_{X/Y} \otimes 1$ is a connection on $\mathscr{E}$ that kills the image of $\Lambda$ in $\mathscr{E}$ and is uniquely determined by this property. When $X$ is $Y$-smooth then $\ker \nabla = \Lambda$ because $\ker \mathrm{d}_{X/Y} = \pi^{-1}\mathscr{O}_Y$ and $\Lambda$ is locally free over $\pi^{-1}\mathscr{O}_Y$.

**Example 1.2.** Assume that $X$ is $Y$-smooth and $\mathscr{E}$ is a vector bundle. Choose an open $U \subseteq X$ such that $\mathscr{E}|_U$ admits a basis $\{e_j\}$ and $U \simeq Y \times V$ for an open $V$ in $\mathbf{C}^n$. We have unique expansions

$$\nabla(e_j) = \sum_{i,k} \Gamma^k_{ij} \mathrm{d}z_i \otimes e_k,$$

where $\{z_i\}$ is the standard coordinate system on $\mathbf{C}^n$ and the $\Gamma^k_{ij}$'s are sections of $\mathscr{O}_X$ over $U$. For a general local section $s = \sum s_j e_j$ of $\mathscr{E}|_U$,

$$\nabla(s) = \sum_{i,k} \left( \frac{\partial s_k}{\partial z_i} + \sum_j \Gamma^k_{ij} s_j \right) \mathrm{d}z_i \otimes e_k.$$

Thus, $\nabla(s) = 0$ is a system of first-order homogeneous linear PDE's. Conversely, given sections $\Gamma^k_{ij}$ of $\mathscr{O}_X$ over $U$ we may use this formula to define a connection on $U$ over $Y$. The $\Gamma^k_{ij}$'s are the *Christoffel symbols* of the connection relative to the local choices of coordinates and basis. In the special case when $X$ is $Y$-smooth with pure relative dimension 1, we write $\Gamma_{ij}$ rather than $\Gamma^1_{ij}$, and $(\Gamma_{ij})$ is the *connection matrix* with respect to the chosen basis of $\mathscr{E}|_U$ and the choice of relative coordinate $z_1$.

A connection is to be viewed as a mechanism for differentiating sections along vector fields: if $\vec{v}$ is a relative vector field on $X$ over $Y$ (*i.e.*, an $\mathscr{O}_Y$-linear derivation of $\mathscr{O}_X$), then the composite $\nabla_{\vec{v}} = (\vec{v} \otimes 1) \circ \nabla : \mathscr{E} \to \mathscr{E}$ satisfies

$$\nabla_{\vec{v}}(fs) = \vec{v}(f) \cdot s + f \cdot \nabla_{\vec{v}}(s).$$

The operation $\nabla_{\vec{v}}$ is "differentiation along the vector field $\vec{v}$." Note that $\vec{v} \mapsto \nabla_{\vec{v}}$ is $\mathscr{O}_X$-linear.

**Remark 1.3.** Let us mention another description of connections that fits with the approach to calculus that is used in crystalline cohomology. The structure sheaf $\mathscr{P}_{X/Y} = \mathscr{O}_{X\times_Y X}/\mathscr{I}_\Delta^2$ of the first infinitesimal neighborhood of the diagonal admits two natural $\mathscr{O}_X$-module structures, the *left-structure* and the *right-structure*; these are defined via pullback along the two projections $X \times_Y X \rightrightarrows X$. The kernel of $\mathscr{P}_{X/Y} \twoheadrightarrow \Delta_*\mathscr{O}_X$ is $\Omega^1_{X/Y}$, and this acquires the same (coherent) $\mathscr{O}_X$-module structure under both the left and right structures. Recall from [5, IV$_4$, 16.3.6] that $\mathrm{d}_{X/Y}(f) = p_2^*(f) - p_1^*(f)$ for local sections $f$ of $\mathscr{O}_X$.

We claim that to specify a connection on a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the same as to specify a $\mathscr{P}_{X/Y}$-linear isomorphism

$$\xi : \mathscr{P}_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{F} \simeq \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{P}_{X/Y}$$

that lifts the identity on $\mathscr{F}$ (working modulo the augmentation ideal $\Omega^1_{X/Y}$); in this map, the tensor product on the left side uses the right-structure on $\mathscr{P}_{X/Y}$ and the tensor product on the right side uses the left-structure on $\mathscr{P}_{X/Y}$. (Note also that it suffices to give $\xi$ merely as a $\mathscr{P}_{X/Y}$-linear map lifting the identity on $\mathscr{F}$, since the coherence of $\mathscr{F}$ forces such a $\xi$ to automatically be an isomorphism.) To link $\xi$ with a connection, observe that the lifting condition on $\xi$ says $\xi(1\otimes s) = s\otimes 1 + \nabla(s)$ for a unique section $\nabla(s)$ of the subsheaf $\mathscr{F}\otimes_{\mathscr{O}_X}\Omega^1_{X/Y}$. The requirement of $\mathscr{P}_{X/Y}$-linearity says $p_2^*(f)\cdot\xi(1\otimes s) = \xi(1\otimes fs)$, or equivalently

$$s\otimes p_2^*(f) + f\cdot\nabla(s) = s\otimes p_1^*(f) + \nabla(fs);$$

since $\mathrm{d}_{X/Y}(f) = p_2^*(f) - p_1^*(f)$, this is exactly the Leibnitz rule. This process may be reversed.

An equivalent description of $\xi$ is that it is a linear isomorphism $q_2^*\mathscr{F} \simeq q_1^*\mathscr{F}$, where the $q_j$'s are the projections to $X$ from the first infinitesimal neighborhood of the diagonal. From the viewpoint of universality, we can restate the data of $\xi$ in another way, as follows. Let $\Delta^{(1)}$ denote the first infinitesimal neighborhood of the diagonal, so there is a closed immersion $\Delta : X \hookrightarrow \Delta^{(1)}$ whose defining ideal has square zero and the maps $q_j : \Delta^{(1)} \to X$ are retractions of $\Delta$. The data $(\Delta^{(1)}, q_1, q_2, \Delta)$ is the final object in the category of 4-tuples $(X', q_1', q_2', \delta)$ consisting of a $Y$-space $X'$ equipped with a square-zero closed $Y$-immersion $\delta : X \hookrightarrow X'$ and a pair of retractions $q_1', q_2' : X' \rightrightarrows X$ (*i.e.*, $q_j'\circ\delta = 1_X$). Thus, we may consider $\xi$ as the functorial specification of isomorphisms ${q_2'}^*\mathscr{F} \simeq {q_1'}^*\mathscr{F}$ for all such 4-tuples $(X', q_1', q_2', \delta)$.

We shall now discuss some basic operations on pairs $(\mathscr{E}, \nabla)$ consisting of a coherent sheaf $\mathscr{E}$ on $X$ and a connection $\nabla$ on $\mathscr{E}$ relative to $Y$. Given $(\mathscr{E}, \nabla)$ and $(\mathscr{E}', \nabla')$, the *direct sum* $\nabla\oplus\nabla'$ on $\mathscr{E}\oplus\mathscr{E}'$ is the map

$$\mathscr{E}\oplus\mathscr{E}' \to (\Omega^1_{X/Y}\otimes_{\mathscr{O}_X}\mathscr{E}) \oplus (\Omega^1_{X/Y}\otimes_{\mathscr{O}_X}\mathscr{E}') \simeq \Omega^1_{X/Y}\otimes_{\mathscr{O}_X}(\mathscr{E}\oplus\mathscr{E}')$$

obtained by forming the direct sum of the maps $\nabla$ and $\nabla'$; this clearly satisfies the Leibnitz rule. On $\mathscr{E}\otimes_{\mathscr{O}_X}\mathscr{E}'$ we define

$$\nabla\otimes\nabla' : s\otimes s' \mapsto \nabla(s)\otimes s' + s\otimes\nabla'(s') \in \Omega^1_{X/Y}\otimes_{\mathscr{O}_X}(\mathscr{E}\otimes_{\mathscr{O}_X}\mathscr{E}');$$

it is straightforward to check that the right side takes the same value when using pairs $(fs, s')$ and $(s, fs')$, so it is well-defined and the resulting map satisfies the Leibnitz rule. This is the *tensor product* connection.

If $\mathscr{E}$ is a vector bundle on $X$ with a connection $\nabla$ relative to $Y$, then the *dual connection* $\nabla^\vee$ on the dual $\mathscr{E}^\vee$ is defined via

$$\nabla^\vee(\ell) = \mathrm{d}_{X/Y} \circ \ell - (1 \otimes \ell) \circ \nabla \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}|_U, \Omega^1_{X/Y}|_U) = \Gamma(U, \Omega^1_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{E}^\vee)$$

for $\ell \in \mathscr{E}^\vee(U)$; the motivation for this is to ensure that the identity

$$\mathrm{d}_{X/Y}(\ell(s)) = (\nabla^\vee(\ell))(s) + (1 \otimes \ell)(\nabla(s))$$

holds for all local sections $s$ of $\mathscr{E}|_U$. The assignment $\ell \mapsto \nabla^\vee(\ell)$ satisfies the Leibnitz rule, so $\nabla^\vee$ is indeed a connection on $\mathscr{E}^\vee$ relative to $Y$. It is not difficult to verify that $\nabla^{\vee\vee} = \nabla$ via the isomorphism $\mathscr{E}^{\vee\vee} \simeq \mathscr{E}$.

**Example 1.4.** For vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ on $X$ equipped with connections $\nabla_1$ and $\nabla_2$ relative to $Y$, the isomorphism $\mathscr{H}om_X(\mathscr{E}_1, \mathscr{E}_2) \simeq \mathscr{E}_2 \otimes \mathscr{E}_1^\vee$ allows us to define a connection $\nabla = \nabla_2 \otimes \nabla_1^\vee$ on this Hom-sheaf. Calculations with elementary tensors show that for a section $\varphi : \mathscr{E}_1|_U \to \mathscr{E}_2|_U$ over an open $U$,

$$\nabla(\varphi) \in \Gamma(U, \mathscr{H}om_X(\mathscr{E}_1, \mathscr{E}_2 \otimes \Omega^1_{X/Y})) = \mathrm{Hom}_U(\mathscr{E}_1|_U, \mathscr{E}_2 \otimes \Omega^1_{X/Y}|_U)$$

is given by $\nabla_2 \circ \varphi - (1 \otimes \varphi) \circ \nabla_1$. In particular, $\nabla(\varphi) = 0$ if and only if $\varphi$ is compatible with the connections $\nabla_1$ and $\nabla_2$, and for $(\mathscr{E}_2, \nabla_2) = (\mathscr{O}_X, \mathrm{d}_{X/Y})$ this recovers the dual connection.

Finally, for any commutative square

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow[h]{} & Y \end{array} \tag{1.1}$$

and $(\mathscr{E}, \nabla)$ on $X$ relative to $Y$, the *pullback* connection $\nabla'$ on $X'$ (relative to $Y'$) is defined as follows. Let $\eta : h'^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ be the canonical map. The map

$$\nabla' : h'^*\mathscr{E} = \mathscr{O}_{X'} \otimes_{h'^{-1}\mathscr{O}_X} h'^{-1}\mathscr{E} \to \Omega^1_{X'/Y'} \otimes_{\mathscr{O}_{X'}} h'^*\mathscr{E}$$

defined by

$$\nabla'(f' \otimes h'^{-1}s) = \mathrm{d}f' \otimes h'^*(s) + f' \cdot (\eta \otimes 1)(h'^*(\nabla(s)))$$

is readily checked to be well-defined (*i.e.*, for any local sections $f'$, $f$, and $s$, the pairs $(f' \cdot h'^{-1}(f), h'^{-1}(s))$ and $(f', h'^{-1}(f) \cdot h'^{-1}(s))$ are sent to the same place), and it satisfies the Leibnitz rule.

The pullback connection is often denoted $h'^*(\nabla)$, though it also depends on the left and bottom sides of (1.1), and it is uniquely characterized by the property that it sends $h'^{-1}(s)$ to $(\eta \otimes 1)(h'^*(\nabla(s)))$. This characterization implies that the pullback operation on connections is naturally transitive and associative with respect to composite pullbacks. Local calculations also show that formation of dual and tensor-product connections is compatible with pullback.

**Example 1.5.** The construction in Example 1.1 is compatible with duality, tensor product, and pullback. To be precise, let $\Lambda$ and $\Lambda'$ be locally free $\pi^{-1}\mathscr{O}_Y$-modules of finite rank, and let $\mathscr{E} = \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} \Lambda$ and $\mathscr{E}' = \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} \Lambda'$. Let $\nabla_\Lambda$ and $\nabla_{\Lambda'}$ be the unique connections on $\mathscr{E}$ and $\mathscr{E}'$ whose kernel contains $\Lambda$ and $\Lambda'$. With respect to the isomorphisms

$$\mathscr{E}^\vee \simeq \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} \Lambda^\vee, \quad \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}' \simeq \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} (\Lambda \otimes_{\pi^{-1}\mathscr{O}_Y} \Lambda'),$$

a local calculation shows that the connections $\nabla_\Lambda^\vee$ and $\nabla_\Lambda \otimes \nabla_{\Lambda'}$ kill the images of $\Lambda^\vee$ and $\Lambda \otimes \Lambda'$, and so uniqueness implies $\nabla_\Lambda^\vee = \nabla_{\Lambda^\vee}$ and $\nabla_\Lambda \otimes \nabla_{\Lambda'} = \nabla_{\Lambda\otimes\Lambda'}$. In a similar manner, for any commutative square as in (1.1), the pullback connection $h'^*(\nabla_\Lambda)$ is identified with $\nabla_{h^*\Lambda}$.

There is a fundamental link between local systems and vector bundles with connection. The starting point is the following reformulation of the local uniqueness of solutions to first-order ODE's with an initial condition:

**Lemma 1.6.** *If $X$ is smooth and $(\mathscr{E}, \nabla)$ is a vector bundle on $X$ with a connection, then for the $\underline{\mathbf{C}}$-module* $\ker \nabla$ *on $X$ the $\mathbf{C}$-linear map $(\ker \nabla)_x \to \mathscr{E}(x)$ is injective for all $x \in X$. In particular, all stalks of* $\ker \nabla$ *are finite-dimensional over $\mathbf{C}$.*

*The sets $X^{\leq d} = \{x \in X \mid \dim(\ker \nabla)_x \leq d\}$ are a locally finite collection of closed sets in $X$, and* $\ker \nabla$ *has locally constant restriction to $X^{\leq d} - X^{\leq (d-1)}$ for all $d$. In particular,* $\ker \nabla$ *is locally constant on $X$ if and only if $x \mapsto \dim(\ker \nabla)_x$ is a locally constant function on $X$.*

In §2 we shall determine when $(\ker \nabla)_x = \mathscr{E}(x)$ for all $x \in X$ (and so in particular $\ker \nabla$ is locally constant).

*Proof.* We may work locally on $X$, so we can assume $X = \mathbf{B}^n$ is a polydisc and $\mathscr{E}$ is globally free. Let $z_1, \dots, z_n$ be the standard coordinates on $\mathbf{B}^n$, and let $\{e_j\}$ be a basis of $\mathscr{E}$. The sections $\mathrm{d}z_i \otimes e_k$ give a basis of $\Omega^1_X \otimes \mathscr{E}$, and so if we expand $\nabla(e_j)$ with respect to this basis we see that the general condition $\nabla(\sum s_i e_i) = 0$ is a system of first-order linear PDE's in the $s_i$'s with coefficients expressed in terms of the $\Gamma^k_{ij}$'s as in Example 1.2. The uniqueness theorem for ODE's with an initial condition may be applied to restrictions along analytic curves to conclude that a section of $\ker \nabla$ over a connected open $U$ is uniquely determined by the specification of its value at a single point $x_0$. Thus, $(\ker \nabla)_x \to \mathscr{E}(x)$ is indeed injective for all $x$ and the sets $X - X^{\leq d}$ are open (so each $X^{\leq d}$ is closed). It is likewise clear that the collection of closed sets $\{X^{\leq d}\}_d$ is locally finite on $X$.

Pick $x \in X^{\leq d} - X^{\leq (d-1)}$, so $d = \dim(\ker \nabla)_x$. By shrinking $X$ around $x$ we may arrange that there is a $d$-dimensional subspace $V \subseteq (\ker \nabla)(X)$ with $V$ mapping isomorphically onto $(\ker \nabla)_x$. It follows that $\mathscr{O}_X \otimes_{\underline{\mathbf{C}}} \underline{V} \to \mathscr{E}$ is a map of vector bundles on the manifold $X$ such that it induces an injection on $x$-fibers. By shrinking $X$ around $x$, we may therefore arrange that this map is a subbundle, and hence the induced map $V = \underline{V}_{x'} \to \mathscr{E}(x')$ is injective for all $x' \in X$. This factors through the subspace $(\ker \nabla)_{x'}$, so the map $\phi : \underline{V} \to \ker \nabla$ induces an injection on stalks at all points. Thus, $\phi$ is a subsheaf inclusion. The pullback of $\phi$ over the subset $X^{\leq d} - X^{\leq (d-1)}$ is therefore an isomorphism on stalks and hence is an isomorphism. Thus, $\ker \nabla$ indeed has locally constant restriction to $X^{\leq d} - X^{\leq (d-1)}$. ■

Let us consider a fundamental example that illustrates the general problems to be considered in what follows. Let $D^* = \{0 < |q| < 1\}$ and let $f : E \to D^*$ be the analytic family of elliptic curves with $q$-fiber $\mathbf{C}^\times / q^{\mathbf{Z}}$; to be rigorous, $E$ is the quotient of $\mathbf{C}^\times \times D^*$ by the action of the $D^*$-group $\mathbf{Z} \times D^\times$ with $(n, q)$ acting by $(t, q) \mapsto (q^n t, q)$. The dual sheaf $\Lambda = (\mathrm{R}^1 f_* \underline{\mathbf{C}})^\vee$ is a local system on $D^*$ whose fiber at $q$ is naturally identified with $\mathrm{H}_1(E_q, \mathbf{C})$ because $f$ is a fibration.

Let $e_1(q_0) \in \mathrm{H}_1(E_{q_0}, \mathbf{C})$ be the pushforward of $(1/2\pi i)\sigma_i \in \mathrm{H}_1(\mathbf{C}^\times, \mathbf{C})$, where $\sigma_i$ is the path in $\mathbf{C}^\times$ that goes once around the origin in the $i$-counterclockwise direction; note that $e_1(q_0)$ is independent of $i$. Let $e_2(q_0) \in \mathrm{H}_1(E_{q_0}, \mathbf{C})$ be the class of the closed loop obtained by pushing forward a choice of path in $\mathbf{C}^\times$ joining 1 to $q_0$; this is only canonical modulo $\mathbf{C}e_1(q_0)$. Let $\gamma_i$ be an $i$-counterclockwise generator of $\pi_1(D^*, q_0)$. For a loop $\gamma$ representing a homotopy class $g$ in $\pi_1(D^*, q_0)$ and any $e \in \mathrm{H}_1(E_{q_0}, \mathbf{C}) = \Lambda_{q_0}$, if we transport the homology class $e$ along $\gamma$ from $\gamma(0) = q_0$ to $\gamma(1) = q_0$ via local constancy of $\Lambda$ (or better: via the *unique* identification $\gamma^*\Lambda \simeq \underline{\Lambda_{q_0}}$ over $[0,1]$ extending the canonical isomorphism $(\gamma^*\Lambda)_0 \simeq \Lambda_{q_0}$ on 0-fibers), then we obtain $g(e)$. Thus, $\gamma_i(e_1(q_0)) = e_1(q_0)$ and $\gamma_i(e_2(q_0))$ is the concatenation of the paths $e_2(q_0)$ and $2\pi i e_1(q_0)$, so $\gamma_i(e_2(q_0)) = e_2(q_0) + 2\pi i e_1(q_0)$.

The non-triviality of the $\pi_1$-action implies that $\Lambda$ does not extend to a local system on $D$. However, and this is the key point, the vector bundle $\mathscr{E} = \mathscr{O}_{D^*} \otimes_{\underline{\mathbf{C}}} \Lambda$ does naturally extend to a rank-2 vector bundle on $D$. The reason this happens is that there exist differential equations on $D^*$ whose local solutions in $\mathscr{O}$ exhibit the same monodromy behavior as $e_2$ when we analytically continue them along $\gamma_i$, and so a suitable $\mathscr{O}$-linear combination kills the non-trivial monodromy from $\Lambda$. More precisely, consider a local solution log to the differential equation $\mathrm{d}f = \mathrm{d}q/q$. If we move around the circle through $q_0$ in the $i$-counterclockwise direction then when we return to $q_0$ we have the local solution $\log + 2\pi i$. Thus, the local section $\log \otimes e_1 - e_2$ is invariant under analytic continuation along a generating loop of $\pi_1$, and so it extends to a global section $v_2$ of $\mathscr{E}$ that is independent of $i$ but is only canonical modulo $\mathbf{C}v_1$. The global sections $v_1 = e_1$ and $v_2$ give a basis of $\mathscr{E}$ over $\mathscr{O}_{D^*}$, but the $\mathbf{C}v_1$-ambiguity in the construction of $v_2$ implies that the intrinsic structure on $\mathscr{E}$ is a short exact sequence

$$0 \to \mathscr{O}_{D^*} \to \mathscr{E} \to \mathscr{O}_{D^*} \to 0$$

with the subobject having basis $v_1$ and the quotient having basis $v_2$ mod $\mathbf{C}v_1$. The choice of $v_2$ splits this sequence.

With respect to the $\mathscr{O}_{D^*}$-basis $\{v_1, v_2\}$, the connection $\nabla = \mathrm{d} \otimes 1$ on $\mathscr{E}$ with kernel $\Lambda$ is given by

$$av_1 + bv_2 \mapsto \frac{\mathrm{d}q}{q} \otimes ((qa' + b)v_1 + qb'v_2)$$

for local sections $a$ and $b$ of $\mathscr{O}_{D^*}$ (with derivatives $a'$ and $b'$), and so when we use the $v_j$'s to extend $\mathscr{E}$ to a free vector bundle $\mathscr{E}' = \mathscr{O}_D^{\oplus 2}$ we see that $\nabla$ extends to a map $\nabla' : \mathscr{E}' \to \Omega^1_D(0) \otimes_{\mathscr{O}_D} \mathscr{E}'$ that is a connection with a simple pole at the origin. Changing $v_2$ modulo $\mathbf{C}v_1$ gives an isomorphic structure of the same type over $D$. The construction of $(\mathscr{E}', \nabla')$ may appear to be *ad hoc*, but we shall prove in §4 that the simple-pole property uniquely characterizes it up to unique isomorphism as an extension of $(\mathscr{E}, \nabla)$ over $D$.

Much of the work that follows is devoted to explaining the general framework for carrying out this type of construction and establishing its functoriality. We will see that the essential property underlying the uniqueness of the preceding construction is the unipotence of the $\pi_1$-action.

## 2. de Rham theory for connections

Just as we uniquely extend the universal $\mathscr{O}_Y$-linear derivation $\mathrm{d}_{X/Y} : \mathscr{O}_X \to \Omega^1_{X/Y}$ to a complex $\Omega^\bullet_{X/Y}$ on the exterior powers of $\Omega^1_{X/Y}$ such that the connecting maps satisfy the Leibnitz rule, we have a version for connections; one new aspect is that it is a non-trivial constraint that we get a complex:

**Theorem 2.1.** *Let $(\mathscr{E}, \nabla)$ be a coherent sheaf on $X$ with a connection $\nabla$ relative to $Y$. There is a unique abelian-sheaf map $\nabla^p : \Omega^p_{X/Y} \otimes \mathscr{E} \to \Omega^{p+1}_{X/Y} \otimes \mathscr{E}$ satisfying $\nabla^p(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^p \omega \wedge \nabla(s)$ for each $p \geq 0$ (so $\nabla^0 = \nabla$), and we have*

$$\nabla^{p+q}((\omega_p \wedge \omega_q) \otimes s) = \nabla^p(\omega_p \otimes s) \wedge \omega_q + (-1)^p \omega_p \wedge \nabla^q(\omega_q \otimes s)$$

*for local sections $\omega_p$ of $\Omega^p_{X/Y}$ and $\omega_q$ of $\Omega^q_{X/Y}$.*

*The composites $\nabla^{p+1} \circ \nabla^p$ vanish for all $p$ if and only if $\nabla^1 \circ \nabla^0 = 0$, and when $X$ is $Y$-smooth then this is equivalent to the general identity $\nabla_{[\vec{v},\vec{w}]} = [\nabla_{\vec{v}}, \nabla_{\vec{w}}]$ for local vector fields $\vec{v}$ and $\vec{w}$ on $X$ relative to $Y$.*

Strictly speaking, $\nabla^p(\omega_p \otimes s) \wedge \omega_q$ is to be computed using the natural local mapping $\Omega^{p+1}_{X/Y} \otimes \mathscr{E} \to \Omega^{p+q+1}_{X/Y} \otimes \mathscr{E}$ defined by $(\cdot) \wedge \omega_q : \Omega^{p+1}_{X/Y} \to \Omega^{p+q+1}_{X/Y}$.

*Proof.* See [4, pp. 9–11] for a proof when $Y = \mathrm{Sp}\,\mathbf{C}$; that proof works *verbatim* in the relative case, and the hypothesis there that the coherent sheaves are vector bundles is not used in the proofs. As with the construction of the de Rham complex, the essential point is existence of the $\nabla^p$'s; uniqueness is obvious.

The relevance of $Y$-smoothness in the final part of the theorem is that in this case we may establish identities among 1-forms by evaluation against relative vector fields since $\Omega^1_{X/Y}$ is its own double-dual. ■

The connection $\nabla$ is *integrable* when $\nabla^1 \circ \nabla^0 = 0$, and the resulting complex $(\Omega^\bullet_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{E}, \nabla^\bullet)$ is the *de Rham complex* of $(\mathscr{E}, \nabla)$.

**Example 2.2.** In Example 1.1, the associated de Rham complex is obtained by applying the functor $\Lambda \otimes_{\pi^{-1}\mathscr{O}_Y} (\cdot)$ to the usual $\mathscr{O}_Y$-linear holomorphic de Rham complex for $X$ over $Y$. Thus, the connections in Example 1.1 are integrable. Integrability is also automatic when $X$ is $Y$-smooth with fibers of dimension at most 1, as then $\Omega^2_{X/Y} = 0$.

It is straightforward to check that formation of the de Rham complex of $(\mathscr{E}, \nabla)$ is compatible with pullback, and that the operations of tensor product, duality, and pullback preserve integrability of connections.

**Remark 2.3.** An alternative description of the integrability condition may be given in the spirit of Remark 1.3, as follows. Identify $\nabla$ with a linear isomorphism $\xi : q_2^*\mathscr{E} \simeq q_1^*\mathscr{E}$ on the first infinitesimal neighborhood of the diagonal in $X \times_Y X$. Let $q_{ij}$ be the projections from the first infinitesimal neighborhood of the triple-diagonal in $X \times_Y X \times_Y X$ to the first infinitesimal neighborhood of the diagonal in $X \times_Y X$. Pulling back $\xi$ along the $q_{ij}$'s defines linear isomorphisms $\xi_{ij} : q_j^*\mathscr{E} \simeq q_i^*\mathscr{E}$. It is natural to ask if the cocycle condition $\xi_{13} = \xi_{12} \circ \xi_{23}$ holds. From the viewpoint of universality, as in Remark 1.3, if we consider $\xi$ as the specification of isomorphisms $\xi_{q_1',q_2'} : {q_2'}^*\mathscr{F} \simeq {q_1'}^*\mathscr{F}$ for all pairs of $Y$-retractions $q_1', q_2' : X' \to X$ of a common square-zero closed $Y$-immersion $\delta : X \hookrightarrow X'$, then the cocycle condition is the transitivity condition

$$\xi_{q_1',q_3'} = \xi_{q_1',q_2'} \circ \xi_{q_2',q_3'}$$

for any triple of retractions of a common $\delta$.

It can be proved that when the cocycle condition holds, then $\nabla$ must be integrable, and that if $X$ is $Y$-smooth then integrability forces the cocycle condition to hold; thus, an integrable connection on coherent sheaf on a smooth $Y$-space may be viewed as first-order descent data. We will not make use of the equivalence of integrability and the cocycle condition in the smooth case, and so we leave it to the interested reader to either carry out the verification or to consult [2, §2] for general arguments presented in the **Q**-scheme setting (or [1, Ch. II, Prop. 3.3.9, Thm. 4.2.11] for proofs in the most general axiomatic framework).

**Example 2.4.** Assume that $X$ is $Y$-smooth and that $\mathscr{E}$ is a vector bundle on $X$. Choose local relative coordinates $z_1, \dots, z_n$ on an open $U$ in $X$, with $U$ sufficiently small so that there exists a basis $\{e_j\}$ of $\mathscr{E}|_U$. A connection $\nabla$ on $\mathscr{E}$ relative to $Y$ is determined by the Christoffel-symbol expansions

$$\nabla_{\partial_{z_i}}(e_j) = \sum_k \Gamma_{ij}^k e_k.$$

We shall now formulate integrability of $\nabla|_U$ as a system of differential equations in the Christoffel symbols.

Since $X$ is $Y$-smooth, by Theorem 2.1 the integrability of $\nabla|_U$ is equivalent to the vanishing of

$$[\nabla_{\vec{v}}, \nabla_{\vec{w}}] - \nabla_{[\vec{v},\vec{w}]} \tag{2.1}$$

for local vector fields $\vec{v}$ and $\vec{w}$ over opens in $U$, and the Leibniz rule implies that it suffices to check such vanishing on a local basis of vector fields. Thus, integrability of $\nabla|_U$ amounts to vanishing of (2.1) when $\vec{v} = \partial_{z_i}$ and $\vec{w} = \partial_{z_j}$ for all $i, j$. Let $\nabla_i = \nabla_{\partial_{z_i}}$. Since $[\partial_{z_i}, \partial_{z_j}] = 0$, we need to compute $\nabla_i \circ \nabla_j - \nabla_j \circ \nabla_i$.

Evaluating on $e_k$ gives

$$\begin{aligned} \nabla_i(\nabla_j(e_k)) - \nabla_j(\nabla_i(e_k)) &= \nabla_i\left(\sum_s \Gamma_{jk}^s e_s\right) - \nabla_j\left(\sum_s \Gamma_{ik}^s e_s\right) \\ &= \sum_\ell \left(\sum_s (\Gamma_{jk}^s \Gamma_{is}^\ell - \Gamma_{ik}^s \Gamma_{js}^\ell) + \left(\frac{\partial \Gamma_{jk}^\ell}{\partial z_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial z_j}\right)\right) e_\ell. \end{aligned}$$

If we let $R_{ijk}^\ell$ denote the coefficient of $e_\ell$ in this sum, then the integrability of $\nabla|_U$ is equivalent to the system of differential equations $R_{ijk}^\ell = 0$. The reader may also check (or see [4, p. 11]) that

$$(\nabla^1 \circ \nabla^0)(e_k) = \sum_\ell \left(\sum_{i<j} R_{ijk}^\ell \, \mathrm{d}z_i \wedge \mathrm{d}z_j\right) \otimes e_\ell,$$

so these $R_{ijk}^\ell$'s describe the operator $\nabla^1 \circ \nabla^0$.

These considerations carry over to the case of $C^\infty$-manifolds, and in the special case when $\mathscr{E} = \mathrm{T}_{X/Y}$ is the tangent bundle, $e_i = \partial_{z_i}$, and $\nabla$ is the Levi-Civita connection associated to a Riemannian metric, the $R_{ijk}^\ell$'s are the coefficients of the *curvature tensor* on a Riemannian manifold. In general, the operator $\nabla^1 \circ \nabla^0$ is called the *curvature* of the connection, since we have seen above that it encodes the obstruction to finding local coordinates $\{z_i\}$ whose associated directional derivatives $\nabla_{\partial_{z_i}}$ commute with each other. (Recall that in multivariable calculus, the vanishing of $\mathrm{d}^2$ is equivalent to the identity $\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}$.)

**Example 2.5.** Continuing the preceding example, let us give geometric formulations of the differential equation $\nabla(s) = 0$ (for $s \in \mathscr{E}(U)$) and the system of differential equations $R_{ijk}^\ell = 0$ that encode integrability of $\nabla|_U$. Let $V = \mathbf{V}_{\mathscr{E}} \to X$ be the geometric vector bundle with sheaf of sections $\mathscr{E}$. Let $\{v_j\}$ be the linear coordinates on $V|_U$ over $U$ that are dual to the chosen basis $\{e_j\}$ of $\mathscr{E}|_U$. Thus, $V|_U$ is coordinatized over $Y$ by the $z_i$'s and $v_j$'s, and so the relative tangent bundle $\mathrm{T}_{V/Y}$ is free over $V|_U$ on the basis of $\partial_{z_i}$'s and $\partial_{v_j}$'s. We may define vector fields

$$X_i = \partial_{z_i} - \sum_k \left( \sum_j v_j \Gamma^k_{ij}(z) \right) \partial_{v_k}$$

on $V|_U$, and a direct calculation shows that

$$[X_i, X_j] = -\sum_{k,\ell} R^\ell_{ijk} v_k \partial_{v_\ell}.$$

Thus, the vanishing of $[X_i, X_j]$ on $V|_U$ is exactly the integrability condition.

The only intervention of the $\partial_{z_j}$'s in $X_i$ is through the term $\partial_{z_i}$, so the $X_i$'s freely generate a vector bundle $\mathscr{W}$ over $V|_U$ that is a subbundle of the part of $\mathrm{T}_{V/Y}$ that lies over $V|_U$, and this subbundle over $V|_U$ has rank equal to the (constant) relative dimension of $X$ over $Y$. The absence of $\partial_{z_k}$'s in the formula for $[X_i, X_j]$ implies that $[X_i, X_j]$ lies in $\mathscr{W}$ if and only if $[X_i, X_j] = 0$. That is, integrability of $\nabla|_U$ is equivalent to the property that the subbundle $\mathscr{W}$ of $\mathrm{T}_{V/Y}|_U$ over $V|_U$ is stable under the bracket on vector fields.

Let $s : U \to V|_U$ be a section in $\mathscr{E}(U)$. The induced map $\mathrm{d}_{X/Y}s : \mathrm{T}_{X/Y}|_U \to \mathrm{T}_{V/Y}|_U$ is characterized by

$$\partial_{z_i} \mapsto \partial_{z_i} + \sum_k \frac{\partial s_k}{\partial z_i} \partial_{v_k}$$

where $s_k = v_k \circ s$, and an inspection of the definition of the $X_j$'s shows that the only possible $\mathscr{O}$-linear relation of the form $\mathrm{d}s(\partial_{z_i}) = \sum_j a_{ij} X_j$ with $a_{ij} \in \mathscr{O}(V|_U)$ is $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = 1$. Thus, $\mathrm{d}s$ factors through the subbundle $\mathscr{W}$ freely spanned by the $X_i$'s if and only if $\mathrm{d}s(\partial_{z_i}) = X_i$; *i.e.*, if and only if

$$\partial_{z_i}(s_k) = -\sum_j s_j \Gamma^k_{ij}$$

for all $i$ and $k$. By Example 1.2, this final system of equations is exactly the condition $\nabla(s) = 0$.

We draw two geometric conclusions in terms of the vector bundle $\mathscr{W}$: a section $s$ of $\mathscr{E}$ over $U$ is killed by $\nabla$ if and only if its differential $\mathrm{T}_{X/Y}|_U \to \mathrm{T}_{V/Y}|_U$ factors through $\mathscr{W}$, and the integrability of $\nabla|_U$ is exactly the condition that $\mathscr{W}$ is stable under the bracket on $\mathrm{T}_{V/Y}|_U$.

The importance of integrability is due to:

**Theorem 2.6** (Riemann–Hilbert correspondence)**.** *Assume $X$ is smooth, and let $(\mathscr{E}, \nabla)$ be a vector bundle on $X$ with an integrable connection. The sheaf* $\ker \nabla$ *is locally constant with $\underline{\mathbf{C}}$-rank equal to the $\mathscr{O}$-rank of $\mathscr{E}$, and the natural map $\mathscr{O}_X \otimes_{\underline{\mathbf{C}}} \ker \nabla \to \mathscr{E}$ carrying $\nabla$ back to $\mathrm{d}_X \otimes 1$ is an isomorphism.*

*The functors $(\mathscr{E}, \nabla) \rightsquigarrow \ker \nabla$ and $\Lambda \rightsquigarrow (\mathscr{O}_X \otimes_{\underline{\mathbf{C}}} \Lambda, \mathrm{d}_X \otimes 1)$ are inverse equivalences of categories between the category of vector bundles on $X$ equipped with an integrable connection and the category of local systems of finite-dimensional* $\mathbf{C}$*-vector spaces on $X$. In particular, the category of such pairs $(\mathscr{E}, \nabla)$ is abelian, with kernels and cokernels formed in the evident manner.*

Combining Example 1.1 and the Riemann–Hilbert correspondence, we see the reason for the terminology "integrability": it is exactly the condition that ensures we can locally (uniquely) solve the differential equation $\nabla(s) = 0$ near any $x_0 \in X$ for an arbitrary initial condition $s(x_0) \in \mathscr{E}(x_0)$. Strictly speaking, the phrase "Riemann–Hilbert correspondence" usually refers to a vast generalization with perverse sheaves replacing local systems and regular holonomic $D$-modules replacing vector bundles with integrable connection.

*Proof.* Once it is proved that $\ker \nabla$ is a local system and that $\alpha : \mathscr{O}_X \otimes_{\underline{\mathbf{C}}} \ker \nabla \to \mathscr{E}$ is always an isomorphism, the abelian-category claim follows from the fact that local systems of finite-dimensional $\mathbf{C}$-vector spaces on $X$ form an abelian category in the evident manner. By Lemma 1.6, $\ker \nabla$ is locally constant if its fiber-ranks are locally constant, and for all $x \in X$ the map $(\ker \nabla)_x \to \mathscr{E}(x)$ is injective. Thus, since a map between vector bundles is an isomorphism if and only if it induces an isomorphism on fibers, it suffices to show that for any $x_0 \in X$ and any $s_0 \in \mathscr{E}(x_0)$, there exists a section $s$ of $\mathscr{E}$ near $x$ with $s(x_0) = s_0$ and $\nabla(s) = 0$. This problem is local near $x_0$, so we may put ourselves in the local situation considered in Example 2.5 with the base $Y$ equal to $\mathrm{Sp}\,\mathbf{C}$. Using notation as is introduced there, the condition $\nabla(s) = 0$ says that $s : U \to \mathbf{V}_{\mathscr{E}}|_U$ has differential factoring through the subbundle $\mathscr{W}$ in the tangent bundle of $\mathbf{V}_{\mathscr{E}}|_U$; moreover, the integrability of $\nabla|_U$ implies that $\mathscr{W}$ is stable under the bracket.

The geometric form of the Frobenius integrability theorem [6, 1.64] says that a subbundle $\mathscr{B}$ of the tangent bundle of a $C^\infty$ (or real-analytic) manifold $M$ is stable under the bracket if and only if for each $m \in M$ there is an open neighborhood $U_m$ of $m$ and a closed connected $C^\infty$-submanifold (or real-analytic submanifold) $N$ in $U_m$ passing through $m$ such that the tangent space $T_n(N) \subseteq T_n(M)$ at each point $n$ is $\mathscr{B}(n)$; the manifold $N$ is called an *integral manifold* for the subbundle $\mathscr{B}$, and it is uniquely determined in $U_m$. This theorem carries over to the complex-analytic case on complex manifolds because a closed $C^\infty$-submanifold $N$ in a complex manifold $M$ is a complex submanifold if and only if $T_n(N)$ is a $\mathbf{C}$-linear subspace of $T_n(M)$ for all $n \in N$ (proof: if $z_1, \dots, z_s$ are local coordinates on $U \subseteq M$ containing $n_0 \in N$ such that $\partial_{z_1}|_{n_0}, \dots, \partial_{z_r}|_{n_0}$ is a basis of the $\mathbf{C}$-subspace $T_{n_0}(N) \subseteq T_{n_0}(M)$, then $\pi : U \to \mathbf{C}^r$ is a submersion at $n_0$ with restriction $\pi_N : U \cap N \to \mathbf{C}^r$ that is a $C^\infty$ open immersion after shrinking $U$. The composite $j$ of $\pi_N^{-1}$ and the inclusion of $U \cap N$ into $M$ is a closed $C^\infty$-embedding of $\pi_N(U \cap N)$ into $U$ with image $U \cap N$; the injective differential of $j$ at each point of $\pi_N(U \cap N)$ is $\mathbf{C}$-linear at each point since $T_n(N)$ is a $\mathbf{C}$-linear subspace of $T_n(M)$ for all $n \in N$, and so $j$ is holomorphic).

Choose an initial condition $s_0 = \sum a_j e_j(x_0) \in \mathscr{E}(x_0)$, and consider this as a point in $\mathbf{V}_{\mathscr{E}}|_U$ over $x_0 \in U$. Let $W$ be the integral manifold for $\mathscr{W}$ through $s_0$ in a small open neighborhood of $s_0$, so $T_w(W) = \mathscr{W}(w)$ for all $w \in W$. The differential of the projection $W \to U$ at $s_0$ sends the basis $\{X_i(s_0)\}$ of $\mathscr{W}(s_0) = T_{s_0}(W)$ to the basis $\{\partial_{z_i}(x_0)\}$ of $T_{x_0}(U)$, and so this projection is a local isomorphism. Its analytic inverse near $x_0$ provides a local section $\sigma : U \to W$ satisfying $\sigma(x_0) = s_0$ after we shrink $U$ around $x_0$, and the composite $s : U \to W \to \mathbf{V}_{\mathscr{E}}|_U$ has differential factoring through $\mathscr{W}$ since $T_w(W) = \mathscr{W}(w)$ for all $w \in W$. Thus, $s \in \mathscr{E}(U)$ is in $\ker \nabla$ and satisfies the initial condition $s(x_0) = s_0$. ■

If $(\mathscr{E}, \nabla)$ is a vector bundle with connection on a manifold, the *flat sections* (or *horizontal sections*) of $\mathscr{E}$ are the sections of the local system $\ker \nabla \subseteq \mathscr{E}$. The

Riemann–Hilbert correspondence says that a vector bundle with integrable connection may be reconstructed from its flat sections. This correspondence is very useful, since local systems are topological and vector bundles with connection require only the language of coherent sheaves (and so are algebraic).

Combining the Riemann–Hilbert correspondence with Example 2.2 and the holomorphic Poincaré lemma, we get:

**Corollary 2.7** (Poincaré lemma for integrable connections)**.** *If* $(\mathscr{E}, \nabla)$ *is a vector bundle with integrable connection on a manifold* $X$*, then the de Rham complex* $(\Omega^\bullet_X \otimes \mathscr{E}, \nabla^\bullet)$ *is a resolution of the local system* $\ker \nabla$*.*

It is a remarkable fact that if $\mathscr{E}$ is a coherent sheaf on a manifold $X$ and $\mathscr{E}$ admits an integrable connection, then $\mathscr{E}$ must be a vector bundle; we will not use this fact. A relativization of (our version of) the Riemann–Hilbert correspondence can also be proved: if $\pi : X \to Y$ is smooth and $\mathscr{E}$ is a coherent sheaf on $X$ admitting an integrable connection $\nabla$ relative to $Y$, then $\ker \nabla$ is a locally free $\pi^{-1}\mathscr{O}_Y$-module and $\mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} \ker \nabla \to \mathscr{E}$ is an isomorphism. In particular, $\mathscr{E}$ is a vector bundle and the category of coherent sheaves on $X$ with integrable connection relative to $Y$ is equivalent to the category of locally free $\pi^{-1}\mathscr{O}_Y$-modules. See [4, 2.23] for an elegant inductive proof of these facts.

## 3. Regular singularities in dimension 1

A very interesting feature of connections is that they can be used to canonically extend a vector bundle across a missing point in the 1-dimensional case. We saw a concrete example worked out at the end of §1, and we wish to study the general problem on an arbitrary complex manifold with pure dimension 1. Since integrability is automatic in the 1-dimensional case, we will not make explicit mention of this condition again; the higher-dimensional and relative generalizations of what we are about to do (see [4]) require integrability.

**Definition 3.1.** Assume $X$ is smooth with pure dimension 1, and let $X^*$ be the complement of a 0-dimensional analytic set $\Sigma$ in $X$, with $j : X^* \hookrightarrow X$ the inclusion. Let $(\mathscr{E}^*, \nabla^*)$ be a vector bundle with connection on $X^*$. This pair has *regular singularities* along $\Sigma$ if there exists a vector bundle $\mathscr{E} \subseteq j_*\mathscr{E}^*$ on $X$ such that $j_*(\nabla^*)$ carries $\mathscr{E}$ into $\Omega^1(\Sigma) \otimes \mathscr{E}$.

In the setting of the definition, the induced map $\nabla : \mathscr{E} \to \Omega^1(\Sigma) \otimes \mathscr{E}$ certainly satisfies the Leibnitz rule, and the pair $(\mathscr{E}, \nabla)$ is also called a vector bundle with connection having regular singularities along $\Sigma$. The definition simply says that $\mathscr{E}^*$ extends to a vector bundle $\mathscr{E}$ on $X$ such that $\nabla^*$ extends to a "meromorphic connection" with at worst simple poles along $\Sigma$. Concretely, the condition on the connection over $X^*$ is that for each any $\sigma \in \Sigma$ there exists a basis of $\mathscr{E}^*$ over a punctured open neighborhood of $\sigma$ such that the connection matrix has entries that are meromorphic at $\sigma$ with at worst a simple pole.

Since pullback along an analytic map of Riemann surfaces carries 1-forms with simple poles to 1-forms with simple poles (proof: $\mathrm{d}(t^e)/t^e = e \cdot \mathrm{d}t/t$), we see that for any analytic map $(X', \Sigma') \to (X, \Sigma)$ we may define pullback for pairs $(\mathscr{E}, \nabla)$ on $X$ with regular singularities along $\Sigma$. Duality and tensor product are defined in the evident manner.

A natural question is whether any pair $(\mathscr{E}^*, \nabla^*)$ on $X^*$ admits an extension to $X$ with regular singularities along $\Sigma$, and whether such an extension is unique. We met an instance of the existence problem in the example at the end of §1. That example rested on constructing differential equations with prescribed local monodromy (*i.e.*, log satisfies $\mathrm{d}f = \mathrm{d}q/q$ and it changes by $2\pi i$ under $i$-counterclockwise transport around the origin), and the general case will also require constructing differential equations whose solutions exhibit prescribed local monodromy. The problem of uniqueness is more subtle; in general, there is only uniqueness in the meromorphic sense:

**Theorem 3.2.** *Any pair $(\mathscr{E}^*, \nabla^*)$ on $X^*$ admits an extension $(\mathscr{E}, \nabla)$ on $X$ with regular singularities along $\Sigma$, and flat sections of $\mathscr{E}^*$ on punctured neighborhoods of any $\sigma \in \Sigma$ are meromorphic at $\sigma$.*

*If $(\mathscr{E}_1, \nabla_1)$ and $(\mathscr{E}_2, \nabla_2)$ are vector bundles on $X$ endowed with connections having regular singularities along $\Sigma$, and $(\mathscr{E}_j^*, \nabla_j^*) = (\mathscr{E}_j, \nabla_j)|_{X^*}$, then any map $\varphi : \mathscr{E}_1^* \to \mathscr{E}_2^*$ that is compatible with the connections is meromorphic along $\Sigma$.*

The absence of essential singularities in flat sections is a crucial fact in the theory, and an analogue holds in the higher-dimensional case.

*Proof.* Let $\mathscr{E} = \mathscr{H}om_X(\mathscr{E}_1, \mathscr{E}_2)$. The identification of $\mathscr{E}$ with $\mathscr{E}_2 \otimes_{\mathscr{O}_X} \mathscr{E}_1^\vee$ allows us to define

$$\nabla = \nabla_2 \otimes \nabla_1^\vee : \mathscr{E} \to \Omega^1_X(\Sigma) \otimes_{\mathscr{O}_X} \mathscr{E}.$$

As in Example 1.4, this procedure identifies $\mathscr{H}om_X((\mathscr{E}_1, \nabla_1), (\mathscr{E}_2, \nabla_2))$ with the kernel of $\nabla$, and so the meromorphicity of $\varphi$ along $\Sigma$ is reduced to the assertion that flat sections of $\mathscr{E}$ over $X^*$ are meromorphic along $\Sigma$. We are now reduced to two local problems: the local existence of regular-singular extensions, and the meromorphicity of flat sections at points of $\Sigma$. We can assume $X = D$ is the open unit disc (with parameter $q$) and $\Sigma = \{0\}$.

**Existence**. Pick $q_0 \in D^*$ and let $\Lambda^* = \ker \nabla^*$. Choose a basis $2\pi i$ of $\mathbf{Z}(1)$, and let $T_i \in \mathrm{Aut}(\Lambda^*_{q_0})$ be the action of the $i$-counterclockwise generator of $\pi_1(D^*, q_0)$. Let $V = \Lambda^*_{q_0}$, and choose $N \in \mathrm{End}(V)$ such that $e^{-2\pi i N} = T_i$; this condition on $N$ is independent of the choice of $i$. It is clear that such an $N$ exists when $T_i$ is near the identity, since the exponential for the Lie group $\mathrm{Aut}(V)$ is étale near the identity. In general, we just have to show that $T$ is a power of an element near the identity. This follows from the multiplicative Jordan decomposition $T = T_{\mathrm{ss}} T_{\mathrm{u}}$ of $T$ as a product of commuting semisimple and unipotent automorphisms of $V$ [3, Cor 1, p. 81], and the fact that $U \mapsto \log(1 + (U-1))$ and $N \mapsto \exp(N)$ set up inverse algebraic bijections between the set of unipotent automorphisms of $V$ and the set of nilpotent endomorphisms of $V$ (carrying powers to multiples).

Let $\mathscr{E}_V = \mathscr{O}_D \otimes_{\underline{\mathbf{C}}} \underline{V}$, and let $\nabla_V : \mathscr{E}_V \to \Omega^1_D(0) \otimes \mathscr{E}_V$ be defined by

$$\nabla_V(f \otimes v) = \mathrm{d}f \otimes v + f \cdot \frac{\mathrm{d}q}{q} \otimes N(v).$$

This is a connection with a regular singularity at the origin. We claim that its restriction to $D^*$ is isomorphic to $(\mathscr{E}^*, \nabla^*)$. By the Riemann–Hilbert equivalence between local systems and vector bundles with (integrable) connection, we just have to prove that the local system $\ker \nabla_V|_{D^*}$ gives rise to a monodromy representation of $\pi_1(D^*, q_0)$ on $\mathscr{E}_{V,q_0} = V$ such that the $i$-counterclockwise generating loop acts by $T_i = e^{-2\pi i N}$. For any $v \in V$, the local section $s(q) = q^{-N}(v) = e^{-(\log q)N}(v)$ of $\mathscr{E}_V$ near $q_0$ (for a local branch of the logarithm) satisfies $\nabla_V(s) = 0$ with initial value $v_0 = q_0^{-N}(v)$ at $q_0$. Analytic continuation of this solution around an $i$-counterclockwise generating loop though $q_0$ gives the value $e^{-2\pi i N}(v_0) = T_i(v_0)$. Since we can find $v$ such that $v_0 = q_0^{-N}(v)$ is any desired element of $V$, the $i$-counterclockwise generating loop of $\pi_1$ acts as $T_i$ on $V$. This settles the existence problem.

**Meromorphicity**. Working locally near the origin of a disc and trivializing $\mathscr{E}$ allows us to suppose $\mathscr{E} = \mathscr{O}_D \otimes_{\underline{\mathbf{C}}} \underline{V}$ for a finite-dimensional $\mathbf{C}$-vector space $V$. Using the basis $\mathrm{d}q/q$ of $\Omega^1_D(0)$, there is a unique

$$N \in \Gamma(D, \mathscr{E}nd(\mathscr{E})) = \mathrm{Hom}_{\underline{\mathbf{C}}}(\underline{V}, \mathscr{E}) = \mathrm{Hom}_{\mathbf{C}}(V, \mathscr{E}(D))$$

such that $\nabla(v) = N(v)\mathrm{d}q/q$ for local sections $v$ of $\underline{V}$; in contrast with the existence step, $N$ might not be constant. We have to prove that if $s \in \mathscr{E}(D^*)$ satisfies $\nabla(s) = 0$, then $s$ is meromorphic at the origin. We may identify $N$ and $s$ with holomorphic maps $N : D \to \mathrm{End}(V)$ and $s : D^* \to V$, and (as in Example 1.2) the condition $\nabla(s) = 0$ means $s'(q) = -(N(q)(s(q)))/q$ on $D^*$.

To prove the meromorphicity of any such $s$, it will be convenient to use a Hermitian inner product $H$ on $V$. Let us briefly recall some constructions with Hermitian inner products. The dual space $V^\vee$ acquires an inner product $H^\vee$ by the requirement $H^\vee(H(\cdot, v), H(\cdot, v')) = H(v', v)$; *i.e.*, the dual-basis to an orthonormal basis is orthonormal. Also, $\mathrm{End}(V) = V \otimes V^\vee$ acquires a Hermitian inner product $H \otimes H^\vee$ by the requirement

$$(H \otimes H^\vee)(v \otimes \ell, v' \otimes \ell') = H(v, v')H^\vee(\ell, \ell');$$

explicitly, $(H \otimes H^\vee)(T, T') = \sum_j H(T(e_j), T'(e_j))$ for any orthonormal basis $\{e_j\}$ of $V$. Let $\|\cdot\|$ denote the associated norms on $V$ and $\mathrm{End}(V)$; it is clear that $\|T(v)\| \leq \|T\| \cdot \|v\|$, since any nonzero $v$ may be scaled to become part of an orthonormal basis.

Choose any $0 < \varepsilon < 1$ and define

$$C = \sup_{|q| \leq \varepsilon} \|N(q)\|, \quad c = \sup_{|q| = \varepsilon} \|s(q)\|.$$

For $0 < |q| \leq \varepsilon$, we shall prove that

$$\|s(q)\| \leq c \left( \frac{|q|}{\varepsilon} \right)^{-C}, \tag{3.1}$$

so $s$ is meromorphic at the origin with $\mathrm{ord}_0(s) \geq -C$.

Pick $q_0$ with $|q_0| = \varepsilon$, and consider the function $\sigma(r) = \|s(rq_0)\|^2$ for $r \in (0, 1]$. We will prove $\sigma(r) \leq \sigma(1) r^{-2C}$. Since $\sigma(1) \leq c^2$, extracting square roots and setting $q = rq_0$ will then give the desired result (3.1) as we vary $r$ over $(0, 1]$ and vary $q_0$ over the circle of radius $\varepsilon$. Using the identity $\sigma(r) = H(s(rq_0), s(rq_0))$, clearly

$$\sigma'(r) = H(\partial_r(s(rq_0)), s(rq_0)) + H(s(rq_0), \partial_r(s(rq_0))),$$

where $\partial_r = \mathrm{d}/\mathrm{d}r$. It is also clear that $\partial_r(s(rq_0)) = q_0 s'(rq_0)$, so

$$\|\partial_r(s(rq_0))\| = \varepsilon \|s'(rq_0)\|.$$

By the Cauchy–Schwarz inequality $|H(v, v')| \leq \|v\| \cdot \|v'\|$ we obtain

$$|\sigma'(r)| \leq 2\|s(rq_0)\| \cdot \|\partial_r(s(rq_0))\| = 2\varepsilon \|s(rq_0)\| \cdot \|s'(rq_0)\|.$$

The equation $s'(q) = -(N(q)(s(q)))/q$ implies $\|s'(q)\| \leq C\|s(q)\|/|q|$, so

$$-\sigma'(r) \leq |\sigma'(r)| \leq \frac{2\varepsilon C}{|rq_0|} \cdot \|s(rq_0)\|^2 = \frac{2C}{r} \cdot \sigma(r).$$

We claim that $\sigma$ has no zeros on $(0,1]$ unless it is identically zero. Suppose $\sigma(r_0)=0$ for some $r_0\in(0,1]$. We may write $\sigma(r)=(r-r_0)^n h(r)$ with $n>0$ and a real-analytic $h$ (if $r_0=1$, note that $\sigma$ extends real-analytically past 1), so

$$\frac{2C}{r}\cdot|r-r_0|^n|h(r)|=\frac{2C}{r}\cdot|\sigma(r)|\geq|\sigma'(r)|=|r-r_0|^{n-1}|nh(r)+(r-r_0)h'(r)|.$$

Cancelling $|r-r_0|^{n-1}$ and evaluating at $r_0$ gives $h(r_0)=0$. Thus, if $\sigma$ vanishes at some point in $(0,1]$ then it has infinite order of vanishing there, and so $\sigma=0$ by real-analyticity.

Since (3.1) for $q=rq_0$ is obvious if $\sigma=0$, it remains to consider the case when $\sigma$ is everywhere positive on $(0,1]$. In this case,

$$(\log\circ\sigma)'=\frac{\sigma'}{\sigma}\geq-\frac{2C}{r}=-(\log(r^{2C}))'.$$

Integration over $[r,1]$ yields

$$\log\left(\frac{\sigma(r)}{\sigma(1)}\right)\geq\log(r^{-2C}).$$

Exponentiation gives the desired estimate. ■

**Definition 3.3.** Let $\mathscr{O}_{X_\Sigma}\subseteq j_*\mathscr{O}_{X^*}$ be the subsheaf of sections that are meromorphic along $\Sigma$. A *meromorphic vector bundle* on $X$ along $\Sigma$ is a locally free sheaf of finite rank over $\mathscr{O}_{X_\Sigma}$.

The functor $\mathscr{O}_{X_\Sigma}\otimes_{\mathscr{O}_X}(\cdot)$ carries holomorphic vector bundles on $X$ to meromorphic vector bundles on $X$ along $\Sigma$, compatibly with tensor products and duality and pullback, and clearly this functor is faithful and essentially surjective. In particular, $\Omega^1_X$ gives rise to a meromorphic vector bundle along $\Sigma$ that we denote $\Omega^1_{X_\Sigma}$. We can use the Leibnitz rule to define the concept of a connection $\nabla:\mathscr{E}\to\Omega^1_{X_\Sigma}\otimes\mathscr{E}$ on a meromorphic vector bundle on $X$ along $\Sigma$. Explicitly, such connections arise from maps $\nabla:\mathscr{E}\to\Omega^1_X(D)\otimes\mathscr{E}$ satisfying the Leibnitz rule, where $\mathscr{E}$ is a holomorphic vector bundle on $X$ and $D$ is an effective divisor supported in $\Sigma$.

Let $\mathrm{MC}_{X_\Sigma}$ be the category whose objects are pairs $(\mathscr{E},\nabla)$ consisting of a holomorphic vector bundle on $X$ endowed with a connection $\nabla$ having regular singularities along $\Sigma$, and whose morphisms are maps of meromorphic vector bundles on $X$ along $\Sigma$ that are compatible with the associated meromorphic connections; note that such a map is automatically holomorphic on the vector bundles over $X^*$ because these bundles are $\mathscr{O}$-free over the locally constant $\underline{\mathbf{C}}$-module of flat sections. We may therefore rephrase Theorem 3.2 as saying that there is an equivalence of categories between local systems on $X^*$ and $\mathrm{MC}_{X_\Sigma}$.

We would like to refine this equivalence by singling out a special class of regular singular connections that are functorially determined by their local systems of flat sections on $X^*$ without the interference of meromorphic vector-bundle maps over $X$. This will clarify the uniqueness of the construction at the end of §1.

## 4. Residues, monodromy, and canonical extensions

In the proof of the existence of regular-singular extensions, there was some freedom of choice of $N$: we only required $e^{-2\pi iN}=T_i$. If $N=N_{\mathrm{ss}}+N_{\mathrm{n}}$ is the additive Jordan decomposition of $N$ as a sum of commuting semisimple and nilpotent endomorphisms, then $e^{-2\pi iN_{\mathrm{ss}}}e^{-2\pi iN_{\mathrm{n}}}$ is the multiplicative Jordan decomposition $T_{i,\mathrm{ss}}T_{i,\mathrm{u}}$ of $T_i$. In particular, the nilpotent $N_{\mathrm{n}}$ is uniquely determined by the local monodromy action $T_i$: it is $-\log(T_{i,\mathrm{u}})/2\pi i$ (this is independent of $i$). However, $N_{\mathrm{ss}}$ is not unique: the only requirement is that it preserves each eigenspace of $T_{i,\mathrm{ss}}$ and acts semisimply on the $\lambda$-eigenspace of $T_{i,\mathrm{ss}}$ with eigenvalues $\mu_k$ satisfying $e^{-2\pi i\mu_k} = \lambda$. There is $\mathbf{Z}$-ambiguity in each $\mu_k$. One way to eliminate this ambiguity is to require $N$ to be nilpotent, but this can only be done when the local monodromy is unipotent.

In order to better understand the meaning of $N$ and its relationship with $T_i$, it is convenient to introduce some more terminology. Fix a basis $2\pi i$ of $\mathbf{Z}(1)$, and choose a small open disc $D \subseteq X$ centered at $\sigma \in \Sigma$ and not meeting any other point of $\Sigma$. Let $D^* = D - \{\sigma\}$. The isomorphism $\mathscr{O}_{X^*} \otimes_{\underline{\mathbf{C}}} \Lambda^* \simeq \mathscr{E}^*$ identifies $\Lambda^*_x$ and $\mathscr{E}^*(x)$ for $x \in X^*$, and so for $x \in D^*$ we get the *monodromy transformation* $T_{x,i} \in \mathrm{Aut}(\mathscr{E}^*(x))$ at $x$; concretely, for $v \in \mathscr{E}^*(x)$, the differential equation $\nabla^*(s) = 0$ with initial condition $s(x) = v$ may be locally uniquely solved along an $i$-counterclockwise generating loop of $\pi_1(D^*, x)$, and this solution specializes to $T_{x,i}(v)$ after going once around the loop. The map $x \mapsto T_{x,i}$ is an analytic section of $\mathscr{E}nd(\mathscr{E}^*)$ over $D^*$ since a solution to a first-order vector-valued ODE has analytic dependence on both initial values and auxiliary parameters (this fact follows from the local construction of solutions via uniform limits of Picard-iterates).

The equivalence between local systems and representations of the fundamental group implies that the monodromy transformation $T_{x,i}$ at one point $x \in D^*$ uniquely determines $(\mathscr{E}^*, \nabla^*)|_{D^*}$, but only up to non-unique isomorphism since the pair $(\mathscr{E}^*_x, T_{x,i})$ admits non-trivial automorphisms. Explicitly, we use $T_{x,i}$ to reconstruct the local system $\Lambda^*|_{D^*}$ (up to non-unique isomorphism) as a representation of $\pi_1(D^*, x)$, where the $i$-counterclockwise generator acts as $T_{x,i}$ on the $x$-fiber, and then $(\mathscr{E}^*, \nabla^*)|_{D^*}$ is associated to this local system as in Example 1.1.

A pair $(\mathscr{E}^*, \nabla^*)$ has *unipotent monodromy* relative to $\Sigma$ when the local monodromy representations $T_{x,i}$ near each $\sigma \in \Sigma$ are unipotent automorphisms; *i.e.*, the fundamental group of a small punctured disc at $\sigma$ acts by unipotent automorphisms on the fiber of the space of flat sections at $x$ near $\sigma$. Let $(\mathscr{E}, \nabla)$ be a holomorphic vector bundle on $X$ endowed with a connection having regular singularities along $\Sigma$, with $(\mathscr{E}^*, \nabla^*)$ its restriction to $X^*$; we have seen that such an $(\mathscr{E}, \nabla)$ may always be found when $(\mathscr{E}^*, \nabla^*)$ is given. Our goal is to single out a preferred $(\mathscr{E}, \nabla)$ when $(\mathscr{E}^*, \nabla^*)$ is given; the criterion for the existence of a preferred extension across $X$ will be given in terms of a differential-equations analogue of the local monodromy transformation, called the *residue* of the regular-singular connection at points of $\Sigma$.

The starting point for the construction of the residue is the observation that for local sections $s$ of $\mathscr{E}$ and $f$ of $\mathscr{O}_X$ near $\sigma \in \Sigma$ such that $f(\sigma) = 0$,

$$\nabla(fs) = \mathrm{d}f \otimes s + f \cdot \nabla(s)$$

is a local section of $\Omega^1_X(\Sigma) \otimes \mathscr{E}$ near $\sigma$ whose image in the fiber at $\sigma$ is killed by the map

$$\Omega^1_X(\Sigma)_\sigma \otimes_{\mathbf{C}} \mathscr{E}(\sigma) \stackrel{\mathrm{res}_\sigma \otimes 1}{\to} \mathscr{E}(\sigma)$$

since the 1-form $\mathrm{d}f$ is holomorphic near $\sigma$ and $f(\sigma) = 0$ (thereby killing any simple pole of $\nabla(s)$ at $\sigma$). Thus, even though $\nabla$ is not $\mathscr{O}_X$-linear, the composite of its induced map on $\sigma$-stalks and the residue-map at $\sigma$ gives a well-defined $\mathbf{C}$-linear map of fibers

$$\mathrm{Res}_\sigma(\nabla) : \mathscr{E}(\sigma) \to \mathscr{E}(\sigma);$$

this $\mathbf{C}$-linear endomorphism of $\mathscr{E}(\sigma)$ is the *residue* of $\nabla$ at $\sigma$.

To make residues explicit, choose a local trivialization $\mathscr{E}|_D \simeq \mathscr{O}_D \otimes_{\underline{\mathbf{C}}} \underline{V}$ over an open disc $D$ around $\sigma$ with coordinate $q$, with $V$ a finite-dimensional $\mathbf{C}$-vector space. We have a holomorphic map $N : D^* \to \mathrm{End}(V)$ characterized by the condition

$$\nabla(fv)(q) = (\mathrm{d}f)(q) \otimes v + \frac{\mathrm{d}q}{q} \otimes f(q) \cdot N(q)(v)$$

for $v \in V$ and a section $f$ of $\mathscr{O}_D$. The residue $\mathrm{Res}_\sigma(\nabla) \in \mathrm{End}(V) = \mathrm{End}(\mathscr{E}(\sigma))$ is $N(0)$. The connection matrix $\Gamma = N/q$ depends on the choice of local trivialization of $\mathscr{E}$ and local coordinate $q$, but its residue $N(0)$ as an element of $\mathrm{End}(\mathscr{E}(\sigma))$ is independent of these choices because we gave an intrinsic construction above.

The relationship between the local residue of a regular-singularities connection and the local monodromy transformation of the associated local system of flat sections is that the residue is a limiting logarithm of the monodromy:

**Theorem 4.1.** *Fix a local trivialization $\mathscr{E}|_D \simeq \mathscr{O}_D \otimes_{\underline{\mathbf{C}}} \underline{V}$ for a finite-dimensional $\mathbf{C}$-vector space $V$. Using the resulting isomorphisms $\mathscr{E}(q) \simeq V$ for $q \in D$, the analytic map $D^* \to \mathrm{End}(V)$ defined by $q \mapsto T_{q,i} \in \mathrm{End}(\mathscr{E}(q)) = \mathrm{End}(V)$ extends analytically across the origin with value $e^{-2\pi i \mathrm{Res}_0(\nabla)} \in \mathrm{End}(\mathscr{E}(0)) = \mathrm{End}(V)$ at the origin.*

*In particular, the characteristic polynomial of $e^{-2\pi i \mathrm{Res}_0(\nabla)}$ is equal to the common characteristic poynomial of the monodromy transformations $T_{x,i}$ for $x \in D^*$, and so the monodromy is unipotent if the residue of the connection has all of its eigenvalues in $\mathbf{Z}$.*

*Proof.* The problem is local; let us formulate it in concrete terms. Choose an analytic map $N : D \to \mathrm{End}(V)$, and for $q_0 \in D^*$ consider the $V$-valued differential equation

$$s'(q) = -\frac{N(q)(s(q))}{q}$$

on $D^*$ with initial condition $s(q_0) = v$. Let $\mathfrak{h}_i \subseteq \mathbf{C} - \mathbf{R}$ be the connected component of our choice of basis $2\pi i$ for $\mathbf{Z}(1)$. Pulling back to $\mathfrak{h}_i$ via $z \mapsto e^{2\pi i z}$ gives the $V$-valued differential equation $f'(z) = -2\pi i \cdot N(e^{2\pi i z})(f(z))$ on $\mathfrak{h}_i$ with initial condition $f(z_0) = v$ for a choice of $z_0$ such that $e^{2\pi i z_0} = q_0$. The unique solution has value $T_{q_0,i}(v) \in V$ at $z_0 + 1$; this is independent of the choice of $z_0$ over $q_0$. We want to understand the behavior of $T_{q,i} \in \mathrm{End}(V)$ for $q$ near 0.

Since $N(q) = N(0) + qN_1(q)$ with $N(0) \in \mathrm{End}(V)$ and $N_1 : D \to \mathrm{End}(V)$ holomorphic, the differential equation for $f$ on $\mathfrak{h}_i$ may be written

$$g_y'(x) = -2\pi i(N(0) + e^{2\pi i z} N_1(e^{2\pi i z}))(g_y(x)) \tag{4.1}$$

where $g_y(x) = f(x + iy) \in V$. The shift $z \mapsto z + 1$ leaves the differential equation invariant, and for $z = x + iy \in \mathfrak{h}_i$ we want to study the operator $T_z : V \to V$ sending $v \in V$ to the value at $x + 1$ of the solution $g_y$ whose value at $x$ is $v$. Since (4.1) encodes analytic continuation of a flat section to an integrable connection (on a punctured disc), and the sheaf of flat sections is locally constant (by the Riemann–Hilbert correspondence), it follows that $g_y$ is either nowhere-vanishing or is identically zero. This strong property of the $g_y$'s ultimately rests on the Frobenius integrability theorem.

Our interest is in the behavior of $T_z(v)$ as $y \to \infty$ and $x$ remains in a bounded interval. More precisely, for the unique solutions $g_y$ to (4.1) (with $z = x + iy$)

on $[-1,1]$ such that $g_y(-1) \in V$ runs through a fixed basis $B$ of $V$, we need to study the relationship between $g_y(x)$ and $T_{x+iy}(g_y(x)) = g_y(x+1)$ for $x \in [-1,0]$; note that as we vary the initial condition $g_y(-1)$ through $B$, the values $g_y(x)$ run through a basis of $V$ for each $x \in [-1,0]$ (due to the linearity of (4.1) and the nowhere-vanishing property for its nonzero solutions that we noted above).

The differential equation for $g_y$ is $g_y'(x) = A(x,y)(g_y(x))$ where

$$A : (x,y) \mapsto A(x,y) = -2\pi i(N(0) + e^{2\pi i(x+iy)}N_1(e^{2\pi i(x+iy)})) \in \mathrm{End}(V)$$

is real-analytic. Since $N_1(q)$ is bounded for $q$ near zero, the exponential decay of $e^{-2\pi y}$ implies that as $y \to \infty$ and $x$ remains bounded, $A$ converges uniformly to the constant map $(x,y) \mapsto -2\pi i N(0)$. It follows that as $y \to \infty$, the solution $g_{y,b} : [-1,1] \to V$ satisfying $g_{y,b}(-1) = b \in B$ converges uniformly to the unique $g_b$ satisfying $g_b'(x) = -2\pi i N(0)(g_b(x))$ with $g_b(-1) = b$.

Since $\{g_{y,b}(x)\}_{b\in B}$ is a basis of $V$ uniformly converging to the basis $\{g_b(x)\}_{b\in B}$ of $V$ as $y \to \infty$ (with $x \in [-1,1]$), it follows that as $y \to \infty$ the operators $T_{x+iy} \in \mathrm{End}(V)$ converge uniformly in $x \in [-1,0]$ to the monodromy operator $T_i$ for the differential equation $g'(x) = -2\pi i N(0)(g(x))$. Hence, $q \mapsto T_{q,i}$ has a removable singularity at $q = 0$ with value $T_i$ at the origin. The equation $g'(x) = -2\pi i N(0)(g(x))$ can be explicitly solved: $g(x) = e^{-2\pi i x N(0)}(v_0)$ for $v_0 \in V$. Thus, $g(x+1) = e^{-2\pi i N(0)}(g(x))$ for any solution $g$ and any $x$, so

$$T_i = e^{-2\pi i N(0)} = e^{-2\pi i \mathrm{Res}_0(\nabla)} \in \mathrm{End}(V).$$

■

The relationship between residues and monodromy shows that if we are given a local system $\Lambda^*$ of finite-dimensional $\mathbf{C}$-vector spaces on $X^*$, and $(\mathscr{E}^*, \nabla^*)$ is the corresponding vector bundle with connection on $X^*$, a regular-singularities extension $(\mathscr{E}, \nabla)$ on $X$ specifies a limiting logarithm of the local monodromy. In general there is no canonical logarithm for an automorphism of a vector space; however, a unipotent automorphism has a unique nilpotent logarithm via the series $\log(1 + (U-1))$ that is polynomial in $U$ for unipotent $U$. Hence, it is reasonable to focus attention on the case when the local monodromy of $\Lambda^*$ at each $\sigma \in \Sigma$ is unipotent; in this case, our proof of existence of a regular-singularities extensions $(\mathscr{E}, \nabla)$ yields an extension with nilpotent residues along $\Sigma$ by choosing the local constant $N$ to be the logarithm of the unipotent local monodromy. Fortunately, the $(\mathscr{E}, \nabla)$ constructed in this way is functorial in $\Lambda^*$:

**Theorem 4.2.** *Let $\mathrm{VC}_X^{\mathrm{nil}}$ be the category of pairs $(\mathscr{E}, \nabla)$ on $X$ consisting of a vector bundle $\mathscr{E}$ on $X$ and a connection $\nabla$ having regular singularities and nilpotent residues along $\Sigma$. Let $\mathrm{LS}_{X^*}^{\mathrm{uni}}$ be the category of local systems of finite-dimensional $\mathbf{C}$-vector spaces on $X^*$ with unipotent local monodromy.*

*The functor $(\mathscr{E}, \nabla) \rightsquigarrow \ker \nabla|_{X^*}$ is an equivalence of categories from $\mathrm{VC}_X^{\mathrm{nil}}$ to $\mathrm{LS}_{X^*}^{\mathrm{uni}}$, and flat sections in $\mathscr{E}(X^*)$ extend holomorphically to $\mathscr{E}(X)$.*

In concrete terms, the final assertion in the theorem says that when the connection has nilpotent residues, any flat section on a punctured disc (centered at a point of $\Sigma$) extends to a holomorphic section of the vector bundle.

*Proof.* We have already seen that the functor does carry $\mathrm{VC}_X^{\mathrm{nil}}$ to $\mathrm{LS}_{X^*}^{\mathrm{uni}}$, and that it is faithful and essentially surjective. The Hom-trick used in the proof of Theorem

3.2 reduces the full faithfulness to the holomorphicity on $X$ for flat sections over $X^*$.

The problem is local, and so we are reduced to considering the following situation. Let $N : D \to \mathrm{End}(V)$ be holomorphic with $N(0)$ nilpotent. We must prove that if $s : D^* \to V$ is holomorphic and $s'(q) = -(N(q)(s(q)))/q$, then $s$ is holomorphic at the origin. Pick a Hermitian inner product on $V$. We will prove that for any solution $s$ on a sector $U$, there is a positive integer $d$ such that $\|s(q)\| = O((\log|q|)^d)$ as $q \to 0$ in $U$; in fact, $d = \dim V$ will suffice. This implies that if $s$ is defined on $D^*$ then $qs(q) \to 0$ as $q \to 0$. Thus, $s$ has a removable singularity at the origin, as desired.

Let $\nabla$ and $\nabla_0$ denote the regular-singular connections on $\mathscr{E} = \mathscr{O}_D \otimes \underline{V}$ given by

$$\nabla(f \otimes v) = \mathrm{d}f \otimes v + f \cdot \frac{\mathrm{d}q}{q} \otimes N(q)(v), \ \ \nabla_0(f \otimes v) = \mathrm{d}f \otimes v + f \cdot \frac{\mathrm{d}q}{q} \otimes N(0)(v).$$

Fix a sector $U$ in $D^*$, and let $\mathbf{e}$ and $\mathbf{e}_0$ denote bases of the associated local systems of flat sections on the simply connected manifold $U$. These give bases of $\mathscr{E}|_U$, due to the equivalence between local systems and vector bundles with integrable connection, and so there is a matrix $M = (m_{ij})$ of holomorphic functions on $U$ such that $e_j = \sum_i m_{ij} e_{0,i}$ for all $j$.

Since

$$\nabla(s) = \nabla_0(s) + \mathrm{d}q \otimes N_1(q)(s)$$

for any section $s$ of $\mathscr{E}$, where $N = N(0) + qN_1$ with $N_1$ holomorphic, we have $\nabla(e_{0,j}) = \mathrm{d}q \otimes N_1(q)(e_{0,j})$ for all $j$. Applying $\nabla$ to the identity $e_j = \sum m_{ij} e_{0,i}$ therefore gives

$$0 = \nabla(e_j) = \sum_i \mathrm{d}m_{ij} \otimes e_{0,i} + \mathrm{d}q \otimes \sum_i m_{ij} N_1(q)(e_{0,i})$$

on $U$. Equivalently,

$$m'_{ij}(q) = -\sum_k m_{kj}(q)\nu_{ik}(q)$$

where $N_1(q)(e_{0,i}(q)) = \sum \nu_{ki}(q) e_{0,k}(q)$. That is, we have the matrix equation $M' = [N_1]_{\mathbf{e}_0} \cdot M$ where $[N_1]_{\mathbf{e}_0}$ has value at $q \in U$ equal to the matrix of $N_1(q) \in \mathrm{End}(V)$ with respect to the basis $\{e_{0,i}(q)\}$ of $V$.

Fix a basis $\mathbf{v}$ of $V$. For $q \in U$, let $P(q)$ be the change-of-basis matrix that converts $\mathbf{e}_0(q)$-coordinates into $\mathbf{v}$-coordinates, so the matrix

$$[N_1]_{\mathbf{v}} = P[N_1]_{\mathbf{e}_0} P^{-1}$$

for $N_1$ with respect to the constant basis $\mathbf{v}$ of $\mathscr{O}_D \otimes_{\underline{\mathbf{C}}} \underline{V}$ is holomorphic on $D$ since $N_1 : D \to \mathrm{End}(V)$ is analytic, and so this matrix is bounded near the origin. Our differential matrix equation on $U$ is

$$M' = P^{-1}[N_1]_{\mathbf{v}} PM.$$

The columns of $P^{-1}$ express the $\mathbf{v}$-coordinates of the $e_{0,i}$'s, and so our general local solution $q^{-N(0)}(v_0)$ in the constant-connection case with nilpotent $N(0)$ shows that each entry of $P^{-1}(q)$ is a polynomial in $\log q$ with degree $\leq d = \dim V$. Thus, the entries in $P^{-1}(q)$ are $O((\log|q|)^d)$ as $q \to 0$ in $U$. Cramer's rule implies a similar conclusion for $P(q)$ with a larger universal exponent replacing $d$. Thus, for a fixed choice of $0 < \varepsilon < 1$ and all $q \in U$ with $0 < |q| \leq \varepsilon \leq 1$ we have

$$\|M'(q)\|_{\mathrm{std}} \leq B(\log(1/|q|))^m \|M(q)\|_{\mathrm{std}} \tag{4.2}$$

for some $B \geq 0$ and some positive integer $m$; we are using the standard norm on $d \times d$ matrices that arises from the standard norm on $\mathbf{C}^d$.

We will now prove that this estimate on $M'$ forces $\|M(q)\|_{\rm std}$ to be bounded on $U$ near the origin, and so the estimate $\|e_{0,i}(q)\| = O((\log|q|)^d)$ as $q \to 0$ in $U$ implies the same for the $\|e_j(q)\|$'s, completing the proof. Fix $0 < \varepsilon < 1$ and let

$$c = \sup_{|q|=\varepsilon, q \in U} \|M(q)\|_{\rm std} < \infty;$$

finiteness is due to the fact that the supremum is taken over a set of $q$'s in $U$ bounded away from the origin (non-compactness of $\{q \in U \,|\, |q| = \varepsilon\}$ is harmless here). Choose $q_0 \in U$ satisfying $|q_0| = \varepsilon$, and define $\sigma(r) = \|M(rq_0)\|_{\rm std}^2$ for positive $r$ in an open interval containing $(0,1]$. Our goal is to show that the real-analytic $\sigma$ on $(0,1]$ is bounded by a positive constant independent of $q_0$.

Arguing exactly as in the proof of Theorem 3.2, but using the estimate (4.2) to replace the role of the estimate $\|s'(q)\| \leq C\|s(q)\|/|q|$ used in the proof of Theorem 3.2, we conclude that

$$-\sigma'(r) \leq |\sigma'(r)| \leq 2B(\log(1/\varepsilon r))^m \sigma(r)$$

and that $\sigma : (0,1] \to \mathbf{R}$ is either identically zero or is everywhere positive. The case $\sigma = 0$ is trivial, so we may assume that $\sigma$ is everywhere positive. Dividing the inequality by $\sigma(r)$ and integrating gives the estimate

$$\log\left(\frac{\sigma(r)}{\sigma(1)}\right) \leq 2B \int_r^1 (\log \frac{1}{\varepsilon t})^m \,\mathrm{d}t = \frac{2B}{\varepsilon} \int_{\varepsilon r}^{\varepsilon} (\log \frac{1}{y})^m \,\mathrm{d}y = \frac{2B}{\varepsilon} \int_{1/\varepsilon}^{1/\varepsilon r} \frac{(\log x)^m}{x^2} \,\mathrm{d}x.$$

Taking $r \to 0^+$ makes the right side increase up to an improper integral that is is finite and independent of $q_0$, so exponentiation gives $\sigma(r) \leq C'\sigma(1) \leq C'c^2$ for some positive constant $C'$ that is independent of $q_0$. ■

We have proved that if $\Lambda^*$ is a local system of finite-dimensional $\mathbf{C}$-vector spaces on $X^*$, and $\Lambda^*$ has unipotent local monodromy along $\Sigma$, then there is a functorially associated vector bundle $\mathscr{E}$ on $X$ extending $\mathscr{E}^* = \mathscr{O}_{X^*} \otimes_{\underline{\mathbf{C}}} \Lambda^*$. The bundle $\mathscr{E}$ is the *canonical extension* (also called the *regular-singularities extension*) of $\mathscr{E}^*$, though of course it also depends on the specification of $\nabla^*$ (as the pair $(\mathscr{E}^*, \nabla^*)$ determines $\Lambda^*$). This implies that the explicit construction with elliptic-curve homology at the end of §1 is not *ad hoc*. As a special case of our general conclusions, unipotent local monodromy along $\Sigma$ is trivial if and only if the connection on the canonical extension has vanishing residues; *i.e.*, the local system $\Lambda^*$ extends (necessarily uniquely and functorially) to a local system $\Lambda$ on $X$ if and only if the canonical extension of $\mathscr{O}_{X^*} \otimes_{\underline{\mathbf{C}}} \Lambda^*$ has a holomorphic connection, and then this canonical extension is $\mathscr{O}_X \otimes_{\underline{\mathbf{C}}} \Lambda$ equipped with its standard holomorphic connection $\mathrm{d}_X \otimes 1$ on $X$.

The canonical extension $\mathscr{E}$ may also be characterized by the property that flat sections of $\mathscr{E}$ and $\mathscr{E}^\vee$ on a sector have logarithmic growth near the origin. There is a higher-dimensional version of all of these results, for local systems having unipotent local monodromy with respect to a normal crossings divisor. See [4, Ch. II, 5.2] for further details.

## References

[1] P. Berthlot, *Cohomologie cristalline des variétés de caractérisque* $p > 0$, Lecture Notes in Mathematics **407**, Springer-Verlag, Berlin, 1974.

[2] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Mathematical Notes **21**, P. U. Press, Princeton, 1978.
[3] A. Borel, *Linear algebraic groups* (2nd ed.), Springer-Verlag, New York, 1991.
[4] P. Deligne, *Equations différentielles à points singuliers réguliers*, Springer–Verlag, New York, 1970.
[5] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique*, Publ. Math. IHES, **4, 8, 11, 17, 20, 24, 28, 32**, (1960-7).
[6] F. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York, 1983.

*E-mail address*: bdconrad@umich.edu