Week 1 Exercises

1. Show if $\varphi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is étale with $\mathrm{char}(k) = 0$, then φ is an isomorphism.

2. Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ by $\varphi(x, y) = (x^2 + y^2, xy - 1)$. Compute the set of points on which φ is étale.

3. Let $\varphi : X \to Y$ be a finite-type morphism. Then show φ is unramified if and only if $\Omega^1_{Y/X} = 0$.

4. Show if $\varphi : X \to Y$ is a finite-type morphism, then it suffices to check φ is unramified at every closed point of x for φ to be unramified. (Q: Can you remove the finite-type assumption?)

5. Let $X \to \mathrm{Spec}(k)$ be étale. Characterize all possible X. (Easier variant: Take X to be connected and affine.)

6. Every étale morphism is locally standard (as defined in the notes).

7. If $X \to Y$ is étale, then it is quasi-finite.

8. Let X be a reduced connected scheme. Suppose $i : Z \to X$ is a closed immersion which is also étale. Show that i is an isomorphism. (Make sure this makes intuitive sense!)

9. Let $\mathcal{O}_K$ be a Dedekind domain with fraction field K. Let L/K be a finite separable field extension, and $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. Then $\mathcal{O}_L$ is a Dedekind domain, and the usual definition for $\mathfrak{p} \in \mathrm{Specm}(\mathcal{O}_K)$ to ramify in $\mathcal{O}_L$ works (i.e. $\mathfrak{p}$ ramifies if $\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_n^{e_n}$ with $e_i > 1$ for some i). Prove only finitely many primes of $\mathcal{O}_K$ ramify in $\mathcal{O}_L$.