Week 1 Exercises - Selected Solutions

(1) Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ be given by $f(T)$. Then the Jacobian criterion implies $f'(T)$ is nonvanishing on $\mathbb{A}^1$, i.e. $f'(T) = c$ for some $c \in k^\times$. As $\mathrm{char}(k) = 0$, this implies $f(T) = cT + a$ for some $a \in k$, i.e. φ is an isomorphism.

Note the condition $\mathrm{char}(k) = 0$ is necessary, as $T \mapsto T^p - T$ is étale but not an isomorphism when $\mathrm{char}(k) = p$. $\square$

(2) We compute the Jacobian to be

$$\begin{vmatrix} 2y & x \\ 2x & y \end{vmatrix} = 2(y^2 - x^2).$$

Thus if $\mathrm{char}(k) = 2$, then the map is nowhere étale. Otherwise, it is étale as long as $y^2 - x^2 \neq 0$, i.e. away from the diagonals. $\square$

(4) By exercise (3), $X \to Y$ is unramified only if the quasicoherent sheaf $\Omega^1_{X/Y}$ vanishes at every point of X. But the vanishing of a quasicoherent sheaf can be checked on closed points (as the vanishing of a module can be checked at maximal ideals). $\square$

(7) Let $\varphi : X \to Y$ be étale and $y \in Y$ a point. Then the scheme theoretic fiber is $X_y := X \times_Y y$. Because étale morphisms are preserved by base change, we have $X_y \to \mathrm{Spec}(k(y))$ is étale. Hence X_y is a finite union of separable extensions of $k(y)$ by (5), and the underlying topological space is finite. $\square$

(Note: It's possible to prove (7) with fewer assumptions. It's already true that an unramified morphism of schemes is quasi-finite.)

(8) If $i : Z \to X$ is an étale closed immersion, then the image of Z is both open and closed, i.e. it is all of X. Because X is reduced, this determines the scheme theoretic image of Z, and i is an isomorphism in the category of schemes (as opposed to just on underlying topological spaces). $\square$

(9) The natural map $\varphi : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$ is generically étale as L/K is finite separable. Thus φ is étale on an open dense subset of $\mathrm{Spec}(\mathcal{O}_L)$, i.e. at all but finitely many points. If $\mathfrak{q}$ is a prime of $\mathcal{O}_L$ lying over $\mathfrak{p}$ in $\mathcal{O}_K$, then φ is unramified at $\mathfrak{q}$ if and only if $\mathcal{O}_{L,\mathfrak{q}}/\mathfrak{p}\mathcal{O}_{L,\mathfrak{q}} \cong \mathcal{O}_L/\mathfrak{q}^e$ is a finite separable extension of $\mathcal{O}_K/\mathfrak{p}$. This clearly only holds if $e(\mathfrak{q}/\mathfrak{p}) = 1$, so all but finitely many primes in $\mathcal{O}_L$ are unramified. $\square$