Week 2 Exercises

1. Compute $\pi_1^{\text{ét}}(X)$ for the schemes below relative to any basepoint:
   (a) $X = \operatorname{Spec}(k)$ for k a field.
   (b) $X = \operatorname{Spec}(\mathbb{Z})$.
   (c) $X = \mathbb{A}^1_k$.
   (d) $X = \mathbb{G}_m$ over a field.
   (e) $X = A$ an abelian variety with a polarization $\lambda : A \to A^\vee$. (Hint: Try when $A = E$ is an elliptic curve as a warm-up.)
   (f) $X = C$ a smooth affine curve over a finite field $\mathbb{F}_q$.

2. Let K be a global field, and S a finite set of places of K. Compute $\pi_1(\operatorname{Spec}(\mathcal{O}_{K,S}))^{ab}$, where $\mathcal{O}_{K,S}$ is the ring of S-units in K.

3. Let X be a connected variety over k which remains connected after extending to k^{sep}. Then show we have an exact sequence
$$1 \to \pi_1(X_{k^{\text{sep}}}, \overline{x}) \to \pi_1(X, \overline{x}) \to G_k \to 1.$$
(Apply to $X = \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ to get an interesting relation to dessins....)

4. Prove Proposition 4.13: if A is a local ring, then $A^h = \lim B$ where $(B, \mathfrak{q})$ is an étale A-algebra with $\mathfrak{q} \cap A = \mathfrak{m}_A$ and $A/\mathfrak{m} \to B/\mathfrak{q}$ is an isomorphism.

5. Prove Puiseux's theorem: if K is an algebraically closed field of characteristic zero, then the only finite extensions of $K(\!(T)\!)$ are of the form $K(\!(T^{1/n})\!)$. In particular, $\pi_1(\operatorname{Spec}(K(\!(T)\!))) = \widehat{\mathbb{Z}}$. (Hint: Use Hensel's lemma on $K[\![T]\!]$.)

6. Compute the strictly local ring of $\mathbb{A}^1$ over an algebraically closed field of characteristic zero at the origin.

7. Show that the functor $B \mapsto B/\mathfrak{m}_A B$ is an equivalence of categories between finite étale A-algebras and finite étale $A/\mathfrak{m}_A$-algebras for A a local Henselian ring. (In particular, if $k_A = A/\mathfrak{m}_A$, then $\pi_1(\operatorname{Spec}(A)) = \pi_1(\operatorname{Spec}(k_A))$.)