Week 2 Exercises - Selected Solutions

(1) (a) By a previous exercise, we know all connected étale covers of $\mathrm{Spec}(k)$ are of the form $\mathrm{Spec}(L)$ where L/k is a finite separable field extension. We can refine this cover by $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ where M/k is Galois, so that $\mathrm{Aut}_k(M) = \mathrm{Gal}(M/k)$. Taking the limit over finite Galois extensions gives $G_k = \mathrm{Gal}(k^s/k)$. □

(e) (Thanks to Mia F. for noticing that the polarization was not necessary for this.) Let $B \to A$ be a finite étale morphism, with B connected. Note that because $B \to A$ is finite and $A \to k$ is projective, we see that B is also a projective variety. Then a theorem about abelian varieties implies that B is also an abelian variety. Using another theorem about abelian varieties, we can find a map $A \to B$ such that the composition $A \to A$ is multiplication by n for some $n \in \mathbb{Z}$. We now compute $\mathrm{Aut}_A(A)$ relative to this map, i.e. automorphisms of A (as an algebraic variety) which preserve the map $[n] : A \to A$.

Suppose $f : A \to A$ is such an isomorphism. Then we know that $f = t_P \circ \varphi$ where φ is an isogeny and t_P is translation by P for some $P \in A$. Because $\varphi(\mathcal{O}) = \mathcal{O}$, the identity for A, we see that $f(\mathcal{O}) = P$ must belong to $A[n]$ to respect the structure morphisms. We claim the map $f \mapsto f(\mathcal{O})$ is an isomorphism of $\mathrm{Aut}_A(A)$ with $A[n]$. First, it is surjective, as t_P is such an automorphism for $P \in A[n]$. For injectivity, suppose $f(\mathcal{O}) = f'(\mathcal{O})$. Then $t_P \circ \varphi(\mathcal{O}) = t_{P'} \circ \varphi'(\mathcal{O})$ implies $P = P'$. But then as t_P is an isomorphism, we have $\varphi = \varphi'$ and $f = f'$.

Hence $\mathrm{Aut}_A(A) \cong A[n]$. Lastly we know that $[n] : A \to A$ is étale if and only if $\mathrm{char}(k) \nmid n$, so taking the limit over all such n we find that $\pi_1(A) \cong T(A)$, where the latter is the "full Tate module" $\prod_{\ell \neq \mathrm{char}(k)} T_\ell(A)$. □

(2) As $\pi_1(\mathrm{Spec}\mathcal{O}_{K,S})^{ab}$ characterizes unramified away from S abelian extensions of K, we know by class field theory that this is the S-class group of K: $h_S(K)$. □

(6) We know that this is $k[[T]] \cap \overline{k(T)}$, but I don't have a better description.... □