Week 3 Exercises - Selected Solutions

(1) Let $(X, \mathcal{T})$ be a topological space. Then the corresponding site $\mathfrak{X}$ has an object for every $U \in \mathcal{T}$ and $Hom(U, V) = \{*\}$ if $U \subseteq V$, and is empty otherwise. The covers are given by usual surjective families of opens $(U_i \to U)_{i \in I}$ with $U = \bigcup_i U_i$.

Let $\mathcal{F}$ be a sheaf on X in the usual sense. Then the functor $\tilde{\mathcal{F}}$ on the site $\mathfrak{X}$ defined by $\tilde{\mathcal{F}}(U) = \mathcal{F}(U)$ and $\tilde{\mathcal{F}}(U \hookrightarrow V) = \mathrm{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ is a presheaf on $\mathfrak{X}$. To see it is a sheaf, let $(U_i \to U)_{i \in I}$ be a cover of U in $\mathfrak{X}$. Then the sheaf condition holds for this cover if and only if

$$\tilde{\mathcal{F}}(U) \to \prod_{i \in I} \tilde{\mathcal{F}}(U_i) \rightrightarrows \prod_{i,j \in I^2} \tilde{\mathcal{F}}(U_i \times_U U_j)$$

is exact. By definition, this is the same as asking that

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I^2} \mathcal{F}(U_i \cap U_j)$$

is exact, as the fiber product of two inclusions is the intersection in the category of open subsets of a topological space. This is equivalent to the usual gluing axiom for a sheaf on an open cover. □

(2) We'll show one direction. Let $\varphi : Y \to X$ be a Galois cover with group G. Then we will show that φ is surjective, finite, and étale.

First, note that we're given $Y \times G := \sqcup_{g \in G} Y \cong Y \times_X Y$ via the map $(y, g) \mapsto (y, yg)$. Hence we have a commutative diagram

$$\begin{array}{ccc} \sqcup_{g \in G} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

Because the map $Y \to X$ is faithfully flat, the properties of being surjective, finite, and étale hold for φ if and only if they hold for the top vertical map. From the description of the identification of the disjoint union with the fiber product (and noting that the top vertical map is said isomorphism composed with projection to the first coordinate), we see that the top vertical map obviously has these properties as it acts as the identity map on connected components. $\square$

(4) Let $(\varphi_i : U_i \to U)_{i \in I}$ be a Zariski cover of U in Sch_X, i.e. $U = \cup_i \varphi_i(U_i)$ and φ_i is an open immersion for all i. We will abbreviate $\mathrm{Hom}_X(A, B)$ by $H(A, B)$ for $A, B \in \mathrm{Sch}_X$. We want to show that

$$H(U, Z) \to \prod_i H(U_i, Z) \rightrightarrows \prod_{i,j} H(U_i \times_U U_j, Z)$$

is exact.

First, suppose we have $f \in H(U, Z)$. Then pulling back under φ_i we get an element of $H(U_i, Z)$ (i.e. restriction to U_i), and it agrees under the next two arrows by commutativity of the diagram:

$$\begin{array}{ccccc} U_i \times U_j & \longrightarrow & U_j & & \\ \downarrow & & \downarrow{\scriptstyle \varphi_j} & & \\ U_i & \xrightarrow{\varphi_i} & U & \longrightarrow & Z. \end{array}$$

Thus $H(U, Z)$ maps to the equalizer.

Now suppose we have a commutative diagram for each i, j:

$$\begin{array}{ccc} U_i \times U_j & \longrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle \varphi_j} \\ U_i & \xrightarrow{\varphi_i} & Z, \end{array}$$

i.e. we have a map from each $U_i \to Z$ which agrees on the fiber products.

We want to create a map $\varphi : U \to Z$. Note $U_{ij} := U_i \cap U_j$ is an open subscheme of U which maps to $U_i \times U_j$ (because it maps to U_i and U_j). Thus the maps $U_i \to Z$ and $U_j \to Z$ agree on the (topological) intersection, so we can glue to get a map (of topological spaces) $\varphi : U \to Z$. To turn this into a map of schemes, we need to show that $\mathcal{O}_Z \to \varphi_* \mathcal{O}_U$ induces a local homomorphism on the stalks. But this must hold because the map on local rings is $\mathcal{O}_{Z,z} \to \mathcal{O}_{U,u} = \mathcal{O}_{U_i,u_i}$ for any U_i with $u_i \to u$ under the open immersion $U_i \to U$ (here we use that $U = \cup U_i$), which is a local homomorphism on the stalks by virtue of being a induced by a morphism of schemes.

Hence we have shown that any element of the first product agreeing on the next two maps arise from a (unique) morphism of schemes $U \to Z$. $\square$