Week 4 Exercises

1. Let $\mathcal{C}$ be a small category. Show the category of (contravariant) functors $\mathcal{F} : \mathcal{C} \to \underline{\text{Ab}}$ is an abelian category.

2. Let $\mathcal{P}$ be a subpresheaf of a sheaf $\mathcal{F}$. Define a presheaf $\mathcal{P}'$ by

   $$\mathcal{P}'(U) = \{s \in \mathcal{F}(U) \mid \exists (U_i \to U) \text{ a covering such that } s|_{U_i} \in \mathcal{P}(U_i)\}.$$

   Show $\mathcal{P}'$ is a subsheaf of $\mathcal{F}$.

3. Show that:

   (a) A functor that admits a left adjoint is left exact.

   (b) A functor that admits a right adjoint is right exact.

   (c) (Ricky) A functor that admits an exact left adjoint preserves injectives.

   (d) (Ricky) A functor that admits a left adjoint preserves (categorical) limits. (Hint: Limits are always a composition of products and equalizers.)

   (e) (Ricky) Note/deduce the dual statements about functors that admit a right adjoint.

4. (Ricky) Show that the forgetful functor $i : \underline{Top} \to \underline{Sets}$ has a left adjoint, but $i : Sch/X \to \underline{Sets}$ does not. (If we think of a left adjoint to a forgetful functor as defining a sort of "free object" on a set, then this is saying there's a natural "free" topological space structure on any set, but not a natural scheme structure. If the "free" analogy doesn't make sense, figure out what the left adjoint for $i : \underline{Groups} \to \underline{Sets}$ should be.)

5. This exercise gives an alternate proof that the category $\text{Sh}(X_{\text{ét}})$ is abelian.

   (a) Show sheafification is left exact (hence exact).

   (b) Prove the following:

   Lemma. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be functors such that:*

   - *$\mathcal{C}$ is additive*

- *$\mathcal{D}$ is abelian*
- *$\mathcal{F}, \mathcal{G}$ are additive*
- *$\mathcal{G}$ is left exact*
- *$\mathcal{F}$ is right adjoint to $\mathcal{G}$*
- *$\mathcal{G} \circ \mathcal{F} \cong \mathrm{id}_{\mathcal{C}}$*

Then $\mathcal{C}$ is abelian.

(c) Show $\mathcal{F} : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{PreSh}(X_{\text{ét}})$ and $\mathcal{G} : \mathrm{PreSh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ being inclusion and sheafification satisfy the conditions of the lemma, and thus $\mathrm{Sh}(X_{\text{ét}})$ is abelian.

6. (Ricky) Show that if $\mathrm{char}(k) \nmid n$, then $0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ is exact for X a variety over k, where the second map is $t \mapsto t^n$.

7. Given $\pi : Y \to X$, show the following.

   (a) The pushforward functor $\pi_* : \mathrm{PreSh}(Y_{\text{ét}}) \to \mathrm{PreSh}(X_{\text{ét}})$ is exact.

   (b) The pushforward functor $\pi_* : \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ is left exact.

8. (Ricky) (Extension by Zero) Let $j : U \to X$ be an open immersion. For $\mathcal{F}$ a sheaf on U_{et}, define $j_!\mathcal{F}$ to be the sheafification of the presheaf:

$$(\varphi : V \to U) \mapsto \begin{cases} \mathcal{F}(V), & \text{if } \varphi(V) \subseteq U, \\ 0, & \text{else} \end{cases}$$

   for $\varphi : V \to U \in U_{et}$.

   (a) Show that $j_!$ is left adjoint to $j^* : \mathrm{Sh}(X_{et}) \to \mathrm{Sh}(U_{et})$. (Hint: The pullback is just restriction to U.)

   (b) Show that the stalks of $j_!\mathcal{F}$ are 0 outside of U (and the usual stalks otherwise). (This explains the name.)

   (c) Deduce that $j_!$ is exact, so that j^* preserves injectives (having an exact left adjoint).

   (d) Let $j_!j^*\mathcal{F} \to \mathcal{F}$ be the canonical map corresponding to the identity in $\mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{F}) \cong \mathrm{Hom}(j_!j^*\mathcal{F}, \mathcal{F})$. Similarly for $\mathcal{F} \to i_*i^*\mathcal{F}$ where $i : Z = X \backslash U \hookrightarrow X$ is the inclusion of the complement. Show that the sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

   is exact.